CE 6303 MECHANICS OF FLUIDS

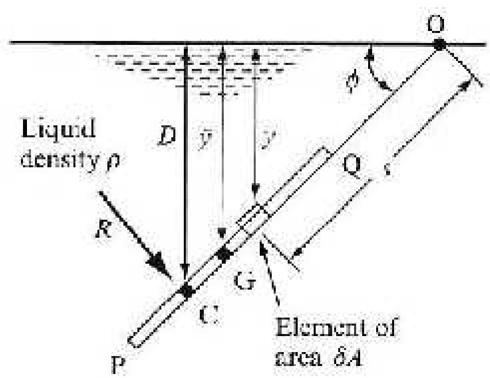
UNIT II FLIUD KINEMATICS AND DYNAMICS

Fluid Kinematics - Flow visualization - lines of flow - types of flow - velocity field and acceleration - continuity equation (one and three dimensional differential forms)- Equation of streamline - stream function - velocity potential function - circulation - flow net. Fluid dynamics - equations of motion - Euler's equation along streamline - Bernoulli's equation – applications - Venturi meter, Orifice meter and Pitot tube. Linear momentum equation and its application.

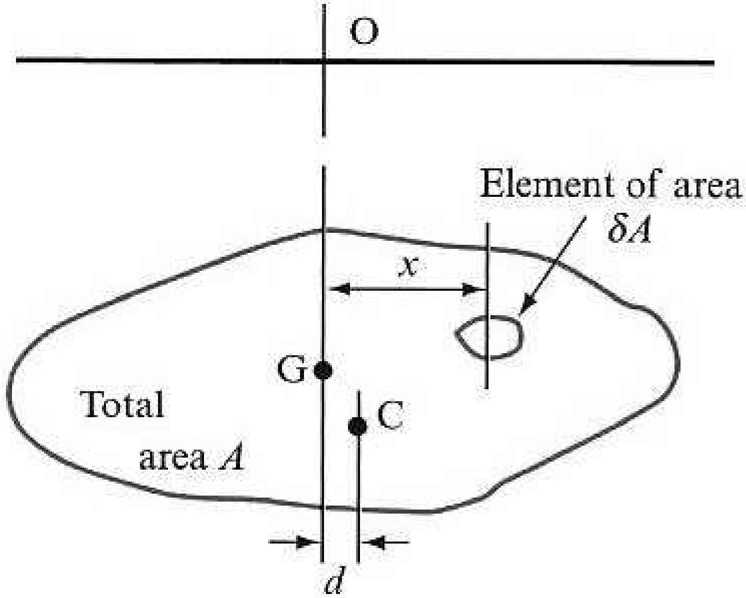
Fluid Action on Surfaces

Plane Surfaces

We consider a plane surface, *PQ*, of area *A*, totally immersed in a liquid of density ρ and inclined at an angle φ to the free surface:



Side Elevation



Front Elevation

If the plane area is symmetrical about the vertical axis OG, then $d = 0$. We will assume that this is normally the case.

Find Resultant Force:

The force acting on the small element of area, δA, is:

$$\delta R = p \cdot \delta A = \rho g y \cdot \delta A$$

The total force acting on the surface is the sum of all such small forces. We can integrate to get the force on the entire area, but remember that y is not constant:

$$R = \int \rho g y \cdot \delta A$$

$$= \rho g \int y \cdot \delta A$$

But $\int y \cdot \delta A$ is just the first moment of area about the surface. Hence:

$$R = \rho g A \bar{y}$$

Where y is the distance to the centroid of the area (point G) from the surface.

Vertical Point Where Resultant Acts:

The resultant force acts perpendicular to the plane and so makes an angle $90° - \phi$ to the horizontal. It also acts through point C, the centre of pressure, a distance D below the free surface. To determine the location of this point we know:

$$\text{Moment of } R \text{ about } O \quad = \quad \begin{array}{l} \text{Sum of moments of forces} \\ \text{on all elements about } O \end{array}$$

Examining a small element first, and since $y = s \sin\phi$, the moment is:

$$\text{Moment of } \delta R \text{ about } O = \left[\rho g \left(s \sin\phi \right) \cdot \delta A \right] s$$

$$= \rho g \sin\phi \left(s^2 \cdot \delta A \right)$$

In which the constants are taken outside the bracket. The total moment is thus:

$$\text{Moment of } R \text{ about } O = \rho g \sin\phi \cdot \int s^2 \cdot \delta A$$

But $\int s^2 \cdot \delta A$ is the second moment of area about point O or just $O\ I$. Hence we have:

$$\text{Moment of } R \text{ about } O = \rho g \sin\phi \cdot I_O$$

$$\rho g A\bar{y} \times OC = \rho g \sin\phi \cdot I_O$$

$$A\bar{y} \times \frac{D}{\sin\phi} = \sin\phi \cdot I_O$$

$$D = \frac{I_O}{A\bar{y}} \cdot \sin^2\phi$$

If we introduce the parallel axis theorem:

$$I_O = I_G + A \times (OC)^2$$

$$= I_G + A \cdot \left(\frac{\bar{y}}{\sin\phi}\right)^2$$

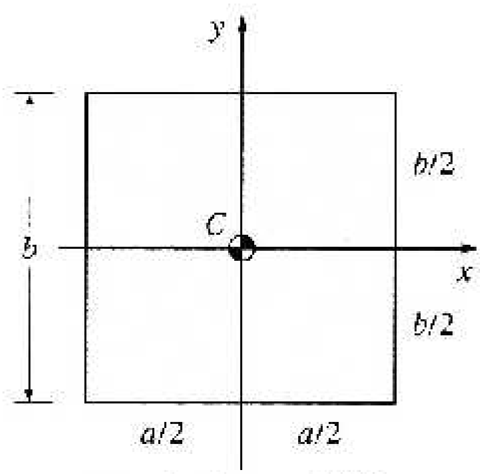
Hence we have:

$$D = \frac{I_G + A\bar{y}^2}{A\bar{y}} \cdot \frac{\sin^2\phi}{\sin^2\phi}$$

$$= y + \frac{I_G}{A\bar{y}}$$

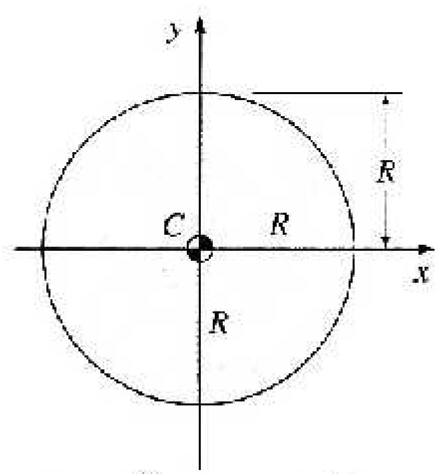
Hence, the centre of pressure, point C, always lies below the centroid of the area, G.

Plane Surface Properties



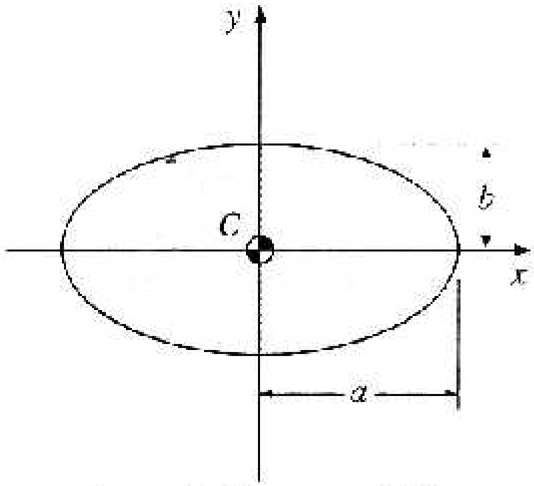
$A = ab,\ I_{xx,C} = ab^3/12$

(*a*) Rectangle



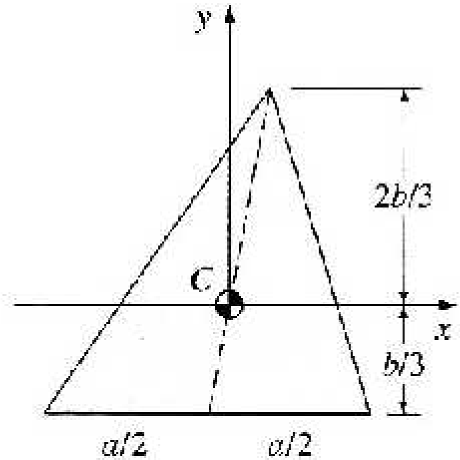
$A = \pi R^2,\ I_{xx,C} = \pi R^4/4$

(*b*) Circle



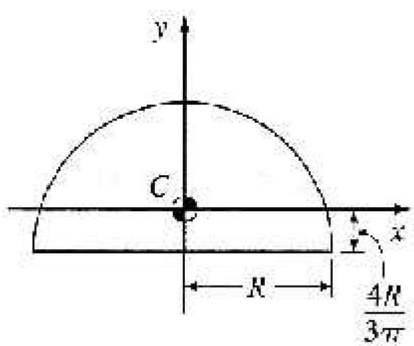
$A = \pi ab,\ I_{xx,C} = \pi ab^3/4$

(c) Ellipse



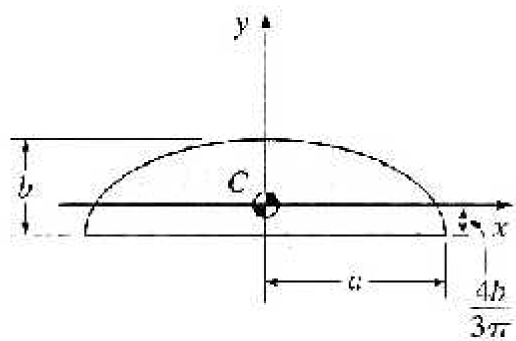
$A = ab/2,\ I_{xx,C} = ab^3/36$

(d) Triangle



$A = \pi R^2/2,\ I_{xx,C} = 0.109757R^4$

(e) Semicircle



$A = \pi ab/2,\ I_{xx,C} = 0.109757ab^3$

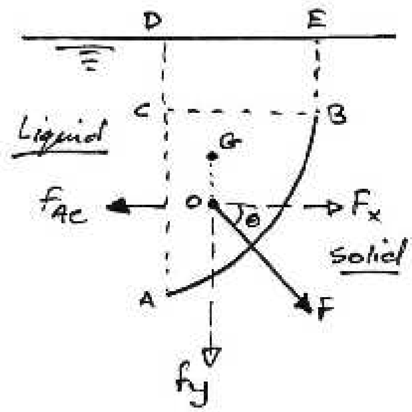
(f) Semiellipse

Curved Surfaces

For curved surfaces the fluid pressure on the infinitesimal areas are not parallel and so must be combined vectorially. It is usual to consider the total horizontal and vertical force components of the resultant.

Surface Containing Liquid

Consider the surface AB which contains liquid as shown below:



Horizontal Component

Using the imaginary plane *ACD* we can immediately see that the horizontal component of force on the surface must balance with the horizontal force *AC F* .

Hence:

$$F_x = \text{Force on projection of surface onto a vertical plane}$$

F must also act at the same level as *F AC* and so it acts through the centre of pressure of the projected surface.

Vertical Component

The vertical component of force on the surface must balance the weight of liquid above the surface. Hence:

$$F_y = \text{Weight of liquid directly above the surface}$$

Also, this component must act through the centre of gravity of the area *ABED*, shown as *G* on the diagram. Resultant

The resultant force is thus:

$$F = \sqrt{F_x^2 + F_y^2}$$

This force acts through the point *O* when the surface is uniform into the page, at an angle of:

$$\theta = \tan^{-1}\frac{F_y}{F_x}$$

to the horizontal. Depending on whether the surface contains or displaces water the angle is measured clockwise (contains) or anticlockwise (displaces) from the horizontal.

KINEMATICS OF FLUIDS

Fluid motion observed in nature, such as the flow of waters in rivers is usually rather chaotic. However, the motion of fluid must conform to the general principles of mechanics. Basic concepts of mechanics are the tools in the study of fluid motion.

Fluid, unlike solids, is composed of particles whose relative motions are not fixed from time to time. Each fluid particle has its own velocity and acceleration at any instant of time. They change both respects to time and space. For a complete description of fluid motion it is necessary to observe the motion of fluid particles at various points in space and at successive instants of time.

Two methods are generally used in describing fluid motion for mathematical analysis, the *Lagrangian* method and the *Eulerian* method.

The Lagrangian method describes the behavior of the individual fluid during its course of motion through space. In rectangular Cartesian coordinate system, Lagrange adopted a, b, c, and t as independent variables. The motion of fluid particle is completely specified if the following equations of motion in three rectangular coordinates are determined:

$$x = F_1(a, b, c, t)$$
$$y = F_2(a, b, c, t)$$
$$z = F_3(a, b, c, t)$$

Eqs. (3.1) describe the exact spatial position (x, y,z) of any fluid particle at different times in terms of its *initial position* (x0 = a, y0 = b, z0 = c) at the given initial time t = t0. They are usually referred to as parametric equations of the path of fluid particles. The attention here is focused on the paths of different fluid particles as time goes on. After the equations describing the paths of fluid particles are determined, the instantaneous velocity components and acceleration components at any instant of time can be determined in the usual manner by taking derivatives with respect to time.

$$u = \frac{dx}{dt} \qquad , a_x = \frac{du}{dt} = \frac{d^2x}{dt^{t}}$$

$$v = \frac{dy}{dt} \qquad , a_y = \frac{dv}{dt} = \frac{d^2y}{dt^2}$$

$$w = \frac{dz}{dt} \qquad , a_z = \frac{dw}{dt} = \frac{d^2z}{dt^2}$$

In which u, v, and w, and ax, ay, and az are respectively the x, y, and z components of velocity and acceleration.

In the Eulerian method, the individual fluid particles are not identified. Instead, a fixed position in space is chosen, and the velocity of particles at this position as a function of time is sought. Mathematically, the velocity of particles at any point in the space can be written,

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

Euler chose x, y, z, and t as independent variables in this method.

The relationship between Eulerian and Lagrangian methods can be shown. According to the Lagrangian method, we have a set of Eqs. (3.2) for each particle which can be combined with Eqs. (3.3) as follows:

$$\frac{dx}{dt} = u(x, y, z, t)$$

$$\frac{dy}{dt} = v(x, y, z, t)$$

$$\frac{dz}{dt} = w(x, y, z, t)$$

The integration of Eqs. (3.4) leads to three constants of integration, which can be considered as initial coordinates a, b, c of the fluid particle. Hence the solutions of Eqs. (3.4) give the equations of Lagrange (Eqs. 3.1). Although the solution of Lagrangian equations yields the complete description of paths of fluid particles, the mathematical difficulty encountered in solving these equations

makes the Lagrangian method impractical. In most fluid mechanics problems, knowledge of the behavior of each particle is not essential. Rather the general state of motion expressed in terms of velocity components of flow and the change of velocity with respect to time at various points in the flow field are of greater practical significance. Therefore the Eulerian method is generally adopted in fluid mechanics. With the Eulerian concept of describing fluid motion, Eqs. (3.3) give a specific velocity field in which the velocity at every point is known. In using the velocity field, and noting that x, y, z are functions of time, we may establish the acceleration components ax, ay, and az by employing the chain rule of partial differentiation,

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

$$u = f_1(x, y, z, t) \quad , \quad a_x = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial t}\right)$$

$$v = f_2(x, y, z, t) \quad , \quad a_y = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial t}\right) \qquad (3.5)$$

$$w = f_3(x, y, z, t) \quad , \quad a_z = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \left(\frac{\partial w}{\partial t}\right)$$

The acceleration of fluid particles in a flow field may be imagined as the superposition of two effects:

1) At a given time t, the field is assumed to become and remain steady. The particle, under such circumstances, is in the process of changing position in this steady field. It is thus undergoing a change in velocity because the velocity at various positions in this field will be different at any time t. This time rate of change of velocity due to changing position in the field is called *convective acceleration*, and is given the first parentheses in the preceding acceleration equations.

2) The term within the second parentheses in the acceleration equations does not arise from the change of particle, but rather from the rate of change of the velocity field itself at the position occupied by the particle at time t. It is called *local acceleration*.

UNIFORM FLOW AND STEADY FLOW

Conditions in a body of fluid can vary from point to point and, at any given point, can vary from one moment of time to the next. Flow is described as *uniform* if the velocity at a given instant is the same in magnitude and direction at every point in the fluid. If, at the given instant, the velocity changes from point to point, the flow is described as *non-uniform*.

A *steady* flow is one in which the velocity and pressure may vary from point to point but do not change with time. If, at a given point, conditions do change with time, the flow is described as *unsteady*.

For example, in the pipe of Fig. 3.1 leading from an infinite reservoir of fixed surfaceelevation, unsteady flow exits while the valve A is being opened or closed; with the valve opening fixed, steady flow occurs under the former condition, pressures, velocities, and the like, vary with time and location; under the latter they may vary only with location.

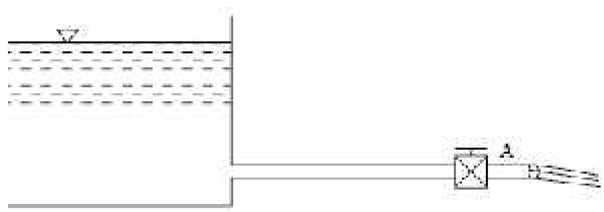


Fig. 3.1

There are, therefore, four possible types of flow.

1) *Steady uniform flow*. Conditions do not change with position or time. The velocity of fluid is the same at each cross-section; e.g. flow of a liquid through a pipe of constant diameter running completely full at constant velocity.

2) Steady non-uniform flow. Conditions change from point to point but not with time. The velocity and cross-sectional area of the stream may vary from crosssection to cross-section, they will not vary with time; e.g. flow of a liquid at a constant rate through a conical pipe running completely full.

3) Unsteady uniform flow. At a given instant of time the velocity at every point is the same, but this velocity will change with time; e.g. accelerating flow of a liquid through a pipe of uniform diameter running full, such as would occur when a pump is started up.

4) Unsteady non-uniform flow. The cross-sectional area and velocity vary from point to point and also change with time; a wave travelling along a channel.

STREAMLINES AND STREAM TUBES

If curves are drawn in a steady flow in such a way that the tangent at any point is in the direction of the velocity vector at that point, such curves are called *streamlines*. Individual fluid particles must travel on paths whose tangent is always in the direction of the fluid velocity at any point. Thus, path lines are the same as streamlines in steady flows.

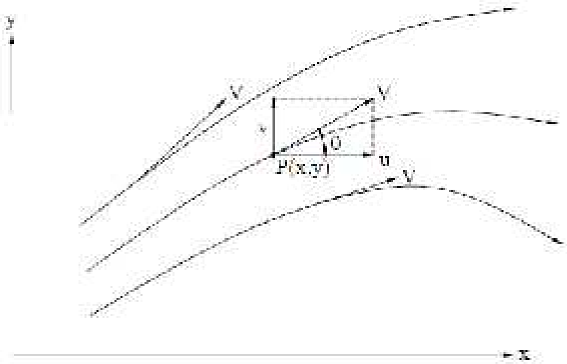


Fig. 3.2

Streamlines for a flow pattern in the xy-plane are shown in Fig. 3.2, in which a streamline passing through the point P (x, y) is tangential to the velocity vector *V*r at P. If u and v are the x and y components of *V* r

$$\frac{v}{u} = Tan\theta = \frac{dy}{dx}$$

Where dy and dx are the y and x components of the differential displacement ds along the streamline in the immediate vicinity of P. Therefore, the differential equation for streamlines in the xy-plane may be written as

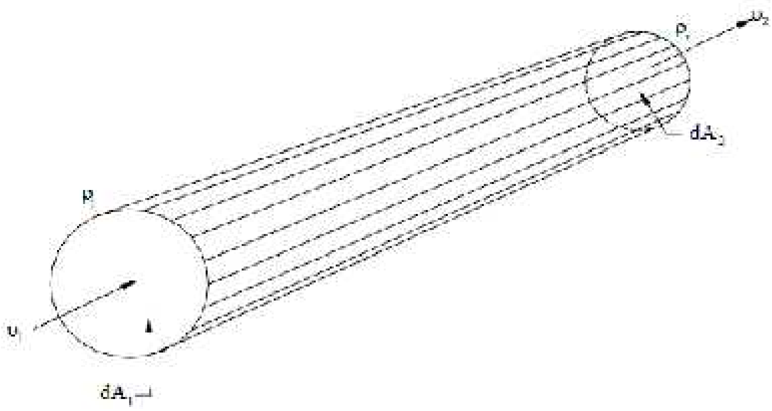
$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad udy - vdx = 0$$

The differential equation for streamlines in space is,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Obviously, a streamline is everywhere tangent to the velocity vector; *there can be no flow occurring across a streamline*. In steady flow the pattern of streamlines remains invariant with time.

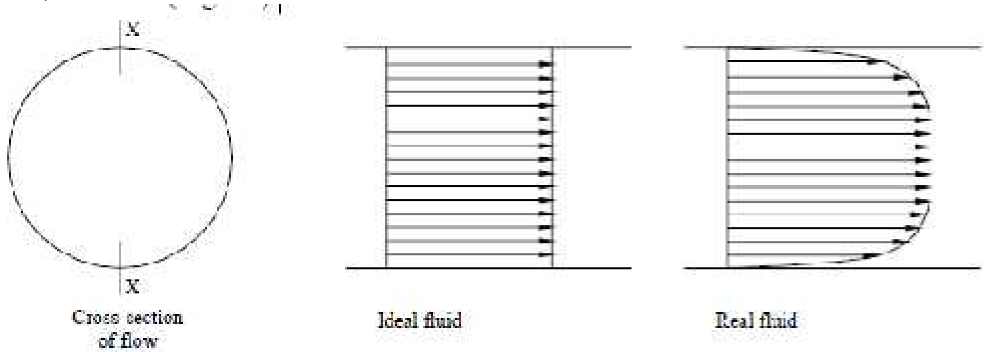
A *stream tube* such as that shown in Fig. 3.3 may be visualized as formed by a bundle of streamlines in a steady flow field. *No flow crosses the wall of a stream tube*. Often times in simpler flow problems, such as fluid flow in conduits, the solid boundaries may serve as the periphery of a stream tube since they satisfy the condition of having no flow crossing the wall of the tube.



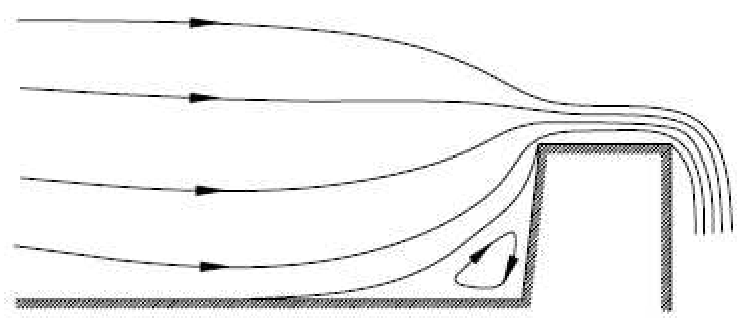
In general, the cross-sectional area may vary along a stream tube since streamlines are generally curvilinear. Only in the steady flow field with uniform velocity will streamlines be straight and parallel. By definition, the velocities of all fluid particles in a uniform flow are the same in both magnitude and direction. If either the magnitude or direction of the velocity changes along any one streamline, the flow is then considered *non-uniform*.

ONE, TWO AND THREE-DIMENSIONAL FLOW

Although, in general, all fluid flow occurs in three dimensions, so that, velocity, pressure and other factors vary with reference to three orthogonal axes, in some problems the major changes occur in two directions or even in only one direction. Changes along the other axis or axes can, in such cases, be ignored without introducing major errors, thus simplifying the analysis. Flow is described as *one-dimensional* if the factors, or parameters, such as velocity,pressure and elevation, describing the flow at a given instant, vary only along the direction of flow and not across the cross-section at any point. If the flow is unsteady, these parameters may vary with time. The one dimension is taken as the distance along the streamline of the flow, even though this may be a curve in space, and the values of velocity, pressure and elevation at each point along this streamline will be the average values across a section normal to the streamline (Fig.3.4).



In *two-dimensional* flow it is assumed that the flow parameters may vary in the direction of flow and in one direction at right angles, so that the streamlines are curves lying in a plane and identical in all planes parallel to this plane.



Thus, the flow over a weir of constant cross-section (Fig.3.5) and infinite width perpendicular to the plane of the diagram can be treated as two-dimensional. In *three-dimensional* flow it is assumed that the flow parameters may vary in space, x in the direction of motion, y and z in the plane of the cross-section.

ENERGY EQUATION:

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as: Consider a stream-line in which flow is taking place in S-direction as shown in figure. Consider a cylindrical element of cross-section dA and length dS. The forces acting on the cylindrical element are:

1.Pressure force pdA in the direction of flow.

2.Pressure force $\left(p + \frac{\partial p}{\partial s} ds \right) dA$ opposite to the direction of flow.

3.Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of S must be equal to the mass of fluid element x acceleration in the S direction.

$$pdA - \left(p + \frac{\partial p}{\partial s} ds \right) dA - \rho g dA dS \cos\theta = \rho dA dS \text{ x } a_s \quad \text{-------------(1)}$$

Where a_s is the acceleration in the direction of S.

Now $a_s = \frac{dv}{dt}$, where v is a function of s and t.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v \partial v}{\partial s} + \frac{\partial v}{\partial t} \qquad \left\{ \frac{ds}{dt} = v \right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$a_s = \frac{v \partial v}{\partial s}$$

Substituting the value of a_s in equation (1) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} dsdA - \rho g dA ds \cos\theta = \rho dA ds \text{ x } \frac{v \partial v}{\partial s}$$

Dividing by $\rho dsdA$, $$-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$$

$$\frac{\partial p}{\rho \partial s} + g\cos\theta + \frac{v\partial v}{\partial s} = 0$$

But from the figure $\cos\theta = \frac{dz}{ds}$

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{dz}{ds} + \frac{v\partial v}{\partial s} = 0 \quad \text{or} \quad \frac{\partial p}{\rho} + gdz + vdv = 0$$

$$\frac{\partial p}{\rho} + gdz + vdv = 0 \quad \text{-----------------(2)}$$

The above equation is known as Euler's equation of motion.

Bernoulli's equation is obtained by integrating the above Euler's equation of motion.

$$\int \frac{\partial p}{\rho} + \int gdz + \int vdv = \text{cons tan t}$$

If the flow is incompressible, ρ is a constant and

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = \text{cons tan t}$$

$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{cons tan t}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{cons tan t} \quad \text{------------(3)}$$

The above equation is known as Bernoulli's equation.

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure Head

$\frac{v^2}{2g}$ = kinetic energy per unit weight or kinetic Head

z= potential energy per unit weight or potential Head

ASSUMPTIONS:

The following are the assumptions made in the derivation of Bernoulli's equation:

(i)The fluid is ideal, i.e. viscosity is zero (ii)The flow is steady

(iii)The flow is incompressible (iv)The flow is irrotational

StatementofBernoulli'sTheorem:

It states in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy. These energies per unit weight of the fluid are:

$$\text{Pressure energy} = \frac{p}{\rho g} \quad \text{Kinetic energy} = \frac{v^2}{2g} \quad \text{Datum energy} = z$$

Thus mathematically, Bernoulli's theorem is written as $\frac{p}{w} + \frac{v^2}{2g} + z = \text{cons tan t}$

RATE OF FLOW OR DISCHARGE (Q):

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(i) For liquids the units of Q are m^3/s or liters/s (ii)For gases the units of Q are kgf/s or Newton/s Consider a fluid flowing flowing through a pipe in which

A= Cross-sectional area of pipe.

V= Average area of fluid across the section

Then discharge $Q = A \times v$

CONTINUITY EQUATION:

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in figure.

Let V_1=Average velocity at cross-section at 1-1

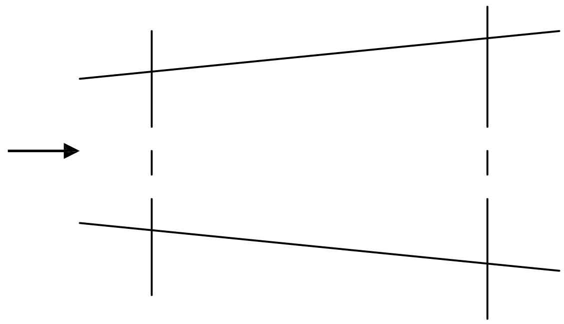
ρ_1 =Density at section 1-1

A_1=Area of pipe at section 1-1

And V_2, ρ_2, A_2 are corresponding values at section 2-2

Then rate of flow at section 1-1 = $V_1 \rho_1 A_1$

Rate of flow at section 2-2 = $V_2 \rho_2 A_2$



According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \text{.....................(1)}$$

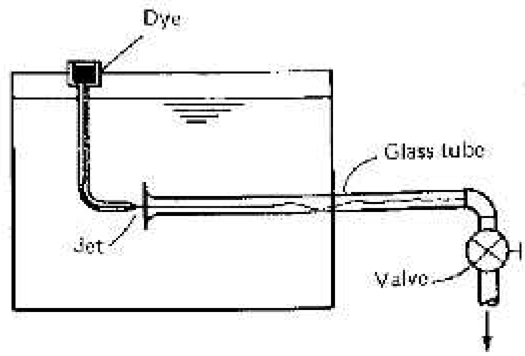
The above equation is applicable to the compressible as well as incompressible fluids is called Continuity Equation. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (1) reduces to

$$A_1 V_1 = A_2 V_2$$

The diameters of a pipe at the sections 1 and 2 are 10cm and 15cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5m/s. Determine the velocity at section 2.

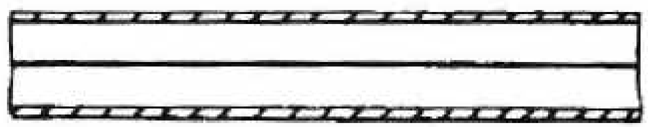
General Concepts

The real behaviour of fluids flowing is well described by an experiment carried out by Reynolds in 1883. He set up the following apparatus:

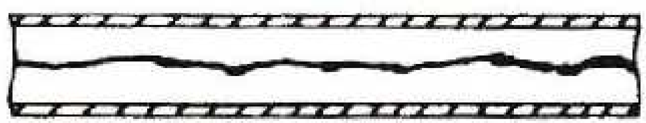


The discharge is controlled by the valve and the small ‘filament’ of dye (practically astreamline) indicates the behaviour of the flow. By changing the flow Reynolds noticed:

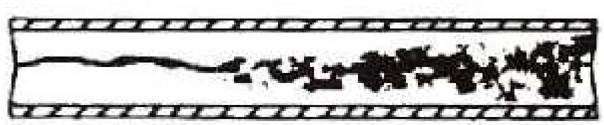
• At low flows/velocities the filament remained intact and almost straight. This type of flow is known as **laminar flow**, and the experiment looks like this:



• At higher flows the filament began to oscillate. This is called **transitional flow** and the experiment looks like:



Lastly, for even higher flows again, the filament is found to break up completely and gets diffused over the full cross-section. This is known as **turbulent flow**:



Reynolds experimented with different fluids, pipes and velocities. Eventually he

found that the following expression predicted which type of flow was found:

$$\mathrm{Re} \quad \frac{\rho \bar{v} l}{\mu}$$

In which Re is called the Reynolds Number; ϱ is the fluid density; *v* is the averagevelocity; *l* is the characteristic length of the system (just the diameter for pipes), and;µ is the fluid viscosity. The Reynolds Number is a ration of forces and hence has nounits.

Flows in pipes normally conform to the following:

• Re < 2000: gives laminar flow;

• 2000 < Re < 4000: transitional flow;

• Re > 4000: turbulent flow.

These values are only a rough guide however. Laminar flows have been found at

Reynolds Numbers far beyond even 4000.

For example, if we consider a garden hose of 15 mm diameter then the limiting average velocity for laminar flow is:

$$\mathrm{Re} = \frac{\rho \bar{v} l}{\mu}$$

$$2000 - \frac{(10^3)\bar{v}(0.015)}{0.55 \times 10^{-3}}$$

$$v = 0.073 \text{ m/s}$$

This is a very low flow and hence we can see that in most applications we deal with turbulent flow. The velocity below which there is no turbulence is called the **critical velocity**.

Characteristics of Flow Types

For laminar flow:

• Re < 2000;

• 'low' velocity;

• Dye does not mix with water;

• Fluid particles move in straight lines;

• Simple mathematical analysis possible;

• Rare in practical water systems.

Transitional flow

• 2000 < Re < 4000

• ‘medium’ velocity

• Filament oscillates and mixes slightly.

Turbulent flow

• Re > 4000;

• ‘high’ velocity;

• Dye mixes rapidly and completely;

• Particle paths completely irregular;

• Average motion is in the direction of the flow;

• Mathematical analysis very difficult - experimental measures are used;

• Most common type of flow.

Background to Pipe Flow Theory

To explain the various pipe flow theories we will follow the historical development of the subject:

Date	Name	Contribution
~1840	Hagen and Poiseuille	Laminar flow equation
1850	Darcy and Weisbach	Turbulent flow equation
1883	Reynolds	Distinction between laminar and turbulent flow
1913	Blasius	Friction factor equation for smooth pipes
1914	Stanton and Pannell	Experimental values of friction factor for smooth pipes
1930	Nikuradse	Experimental values of friction factor for artificially rough pipes
1930s	Prandtl and von Karman	Equations for rough and smooth friction factors
1937	Colebrook and White	Experimental values of the friction factor for commercial pipes and the transition formula
1944	Moody	The Moody diagram for commercial pipes
1958	Ackers	Hydraulics Research Station charts and tables for the design of pipes and channels
1975	Barr	Solution of the Colebrook-White equation

Laminar Flow

Steady Uniform Flow in a Pipe: Momentum Equation

The development that follows forms the basis of the flow theories applied to laminar flows. We remember from before that at the boundary of the pipe, the fluid velocity is zero, and the maximum velocity occurs at the centre of the pipe. This is because of the effect of viscosity. Therefore, at a given radius from the centre of the pipe the velocity is the same and so we consider an elemental annulus of fluid:

FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE:

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through circular pipe will be viscous or laminar, if the Reynold's number is less than 2000. The expression for Reynold's number is given by

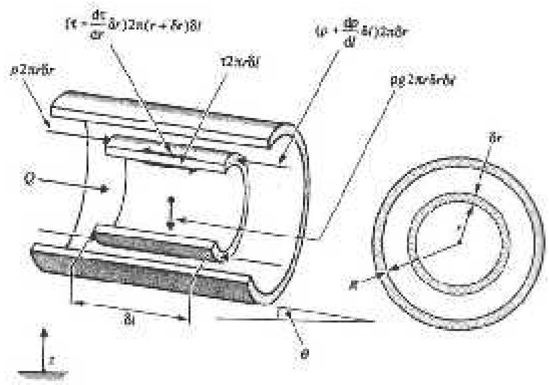
$$R_e = \frac{\rho v d}{\mu}$$

Where ϱ= Density of fluid flowing through pipe,

V = Average velocity of fluid,

D = Diameter of pipe and,

μ = Viscosity of fluid



Consider a horizontal pipe of radius R. The viscous fluid is flowing from left to right in the pipe as shown in figure. Consider a fluid element of radius r, sliding in a cylindrical fluid element of radius (r+dr). Let the length of fluid element be Δx. If 'p' is the intensity of pressure on the face AB, then the intensity of pressure on the face CD will be $\left(p+\frac{\partial p}{\partial x}\Delta x\right)$. The the forces acting on the fluid element are:

1. The pressure force, p x πr^2 on face AB

2. The pressure force $\left(p+\frac{\partial p}{\partial x}\Delta x\right).\pi r^2$ on face CD

3. The shear force, $\tau\times 2\pi r\Delta x$ on the surface of fluid element. As there is no acceleration, hence the summation of all forces in the direction of flow must be zero.

$$p\ \pi r^2 - \left(p+\frac{\partial p}{\partial x}\Delta x\right).\pi r^2 - \tau\times 2\pi r\Delta x = 0$$

$$-\frac{\partial p}{\partial x}\Delta x\pi r^2 - \tau\times 2\pi r\Delta x = 0$$

$$-\frac{\partial p}{\partial x}r - 2\tau = 0$$

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \quad \text{-------------------(1)}$$

The shear stress τ across a section varies with 'r' as $\frac{\partial p}{\partial x}$ across a section is constant. Hence shear stress across a section is linear as shown in figure.

(i) Velocity Distribution: To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu\frac{\partial u}{\partial y}$ is substituted in equation (1)

But in the relation $\tau = \mu\frac{\partial u}{\partial y}$, y is measured from the pipe wall. Hence

$$y = R - r \qquad \text{and} \quad dy = -\,dr$$

$$\tau = \mu \frac{\partial u}{-\partial r} = -\mu \frac{du}{dr}$$

Substituting this value in equation (1)

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x} \frac{r}{2}$$

$$\frac{du}{dr} = \frac{1}{2\mu} \frac{\partial p}{\partial x} r$$

Integrating the equation w.r.t 'r' we get

$$u = \frac{1}{4\mu} \frac{\partial p}{\partial x} r^2 + C \text{ ------------- (2)}$$

Where C is the constant of integration and its value is obtained from the boundary condition that at r=R, u=0

$$0 = \frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 + C$$

$$C = -\frac{1}{4\mu} \frac{\partial p}{\partial x} r^2$$

Substituting this value of C in equation (2), we get

$$u = \frac{1}{4\mu} \frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu} \frac{\partial p}{\partial x} R^2$$

$$u = \frac{1}{4\mu} \frac{\partial p}{\partial x} \left[R^2 - r^2\right] \text{ ----------------------- (3)}$$

In equation (3) values of μ , $\frac{\partial p}{\partial x}$ and r are constant, which means the velocity u, varies with the square of r. Thus the equation (3) is a equation of parabola. This shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in fig.

(ii) Ratio of Maximum velocity to average velocity:

The velocity is maximum, when r =0 in equation (3). Thus maximum velocity, U_{max} is obtained as

$$U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 \text{ ------------------- (4)}$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge of the fluid across the section by the area of the pipe $\left(\pi R^2\right)$. The discharge (Q) across the section is obtained by considering the through a ring element of radius r and thickness the as shown in fig (b). The fluid flowing per second through the elementary ring

dQ= velocity at a radius r x area of ring element

$$= u \times 2\pi r dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right] \times 2\pi r dr$$

$$Q = \int_o^R dQ = \int_0^R = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right] \times 2\pi r dr$$

$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi \int_0^R \left(R^2 - r^2\right) r dr$$

$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi \int_0^R \left(R^2 r - r^3\right) dr$$

$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^4}{4}\right] = \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right) \times 2\pi R^4$$

$$\text{Average velocity, } \bar{u} = \frac{Q}{Area} = \frac{\frac{\pi}{8\mu =}\left(\frac{-\partial p}{\partial x}\right)R^4}{\pi R^2}$$

$$\bar{u} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 \text{ --------------- (5)}$$

Dividing equation (4) by equation (5)

$$\frac{U_{max}}{\bar{u}} = \frac{\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2}{\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2} = 2.0$$

Ratio of maximum velocity to average velocity = 2.0

(iii) Drop of pressure for a given length (L) of a pipe:

From equation (5), we have

$$\bar{u} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 \qquad \text{or} \qquad \left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}$$

Integrating the above equation w.r.t . x, we get

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2}dx$$

$$-[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_1 - x_1]$$

$$[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_2 - x_1]$$

$$= \frac{8\mu\bar{u}}{R^2}L \qquad \{x_2 - x_1 = L \text{ from equation (3)}\}$$

$$= \frac{8\mu\bar{u}L}{(D/2)^2} \qquad \{R = \frac{D}{2}\}$$

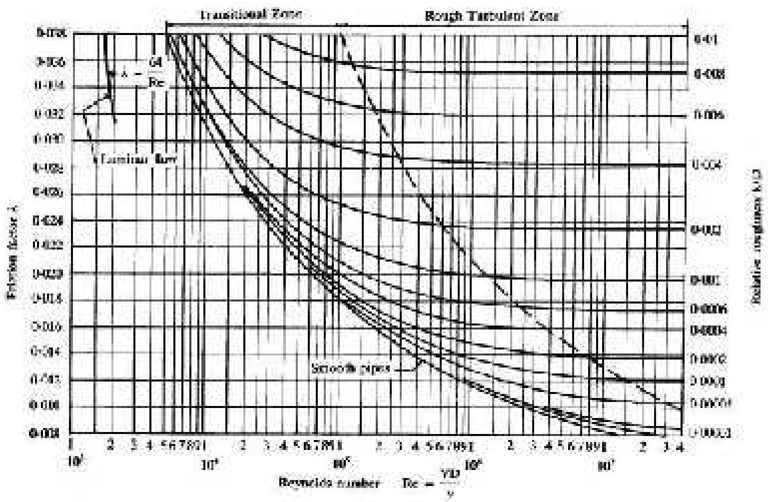
$$[p_1 - p_2] = \frac{32\mu\bar{u}L}{D^2},$$ Where $p_1 - p_2$ is the drop of pressure

Loss of pressure head = $\frac{p_1 - p_2}{\rho g}$

$$\frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \qquad \text{---------------------- (6)}$$

Equation (6) is called Hagen Poiseuille Formula.

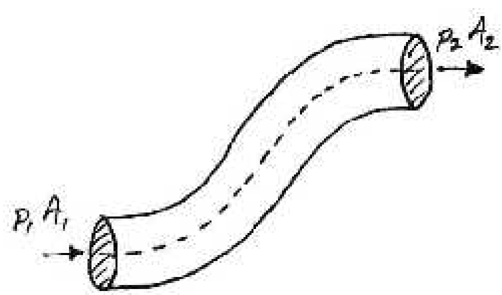
- **DARCY – WEISBACH EQUATION (Derivation refer class notes)**
- **Moody diagram for friction factor:**



The Momentum Equation

Development

We consider again a general streamtube:



In a given time interval, δt , we have:

$$\text{momentum entering} = \rho Q_1 \delta t\, v_1$$
$$\text{momentum leaving} = \rho Q_2 \delta t\, v_2$$

From continuity we know 1 2 $Q = Q = Q$. Thus the force required giving the change in momentum between the entry and exit is, from Newton’s Second Law:

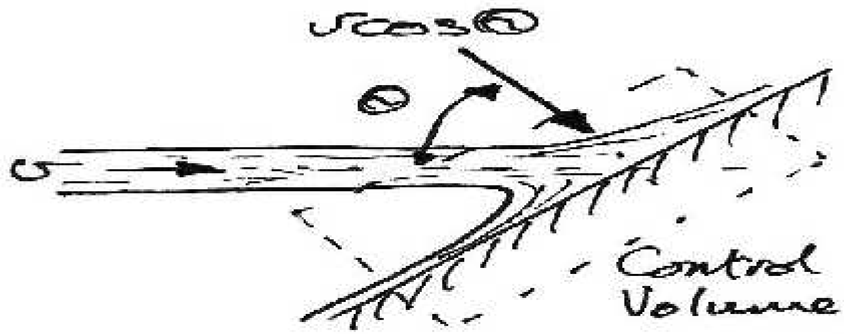
$$F = \frac{d(mv)}{dt}$$

$$F = \frac{\rho Q \delta t (v_2 - v_1)}{\delta t} = \rho Q (v_2 - v_1)$$

This is the force acting on a fluid element in the direction of motion. The fluid exerts an equal but opposite reaction to its surroundings.

Application – Fluid Striking a Flat Surface

Consider the jet of fluid striking the surface as shown:



The velocity of the fluid normal to the surface is:

$$v_{normal} = v \cos\theta$$

This must be zero since there is no relative motion at the surface. This then is also the change in velocity that occurs normal to the surface. Also, the mass flow entering the control volume is:

$$\rho Q = \rho A v$$

Hence:

$$F = \frac{d(mv)}{dt} = (\rho A v)(v\cos\theta) = \rho A v^2 \cos\theta$$

And if the plate is perpendicular to the flow then:

$$F = \rho A v^2$$

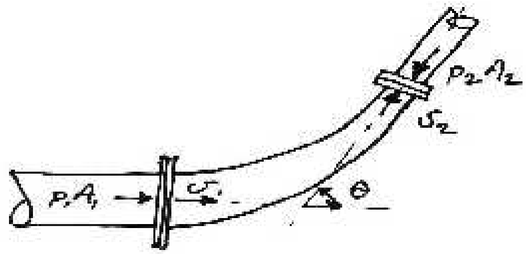
Notice that the force exerted by the fluid on the surface is proportional to the velocity squared. This is important for wind loading on buildings. For example, the old wind loading code

$$q = 0.613 v_s^2 \qquad (\text{N/m}^2)$$

In which v_s is the design wind speed read from maps and modified to take account of relevant factors such as location and surroundings.

Application – Flow around a bend in a pipe

Consider the flow around the bend shown below. We neglect changes in elevation and consider the control volume as the fluid between the two pipe joins.



The net external force on the control volume fluid in the x-direction is:

$$p_1 A_1 - p_2 A_2 \cos\theta + F_x$$

In which F_x is the force on the fluid by the pipe bend (making it 'go around the corner'). The above net force must be equal to the change in momentum, which is:

$$\rho Q (v_2 \cos\theta - v_1)$$

Hence:

$$p_1 A_1 - p_2 A_2 \cos\theta + F_x = \rho Q (v_2 \cos\theta - v_1)$$

$$F_x = \rho Q (v_2 \cos\theta - v_1) - p_1 A_1 + p_2 A_2 \cos\theta$$

$$= (\rho Q v_2 + p_2 A_2)\cos\theta - (\rho Q v_1 + p_1 A_1)$$

Similarly, for the y-direction we have:

$$-p_2 A_2 \sin\theta + F_y = \rho Q (v_2 \sin\theta - 0)$$

$$F_y = \rho Q (v_2 \sin\theta - 0) + p_2 A_2 \sin\theta$$

$$= (\rho Q v_2 + p_2 A_2)\sin\theta$$

The resultant is:

$$F = \sqrt{F_x^2 + F_y^2}$$

And which acts at an angle of:

$$\theta = \tan^{-1} \frac{F_y}{F_x}$$

This is the force and direction of the bend on the fluid. The bend itself must then be supported for this force. In practice a manhole is built at a bend, or else a thrust block is used to support the pipe bend.

PROBLEM -1

The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 lit/sec. the section 1 is 6m above datum. If the pressure at section 2 is 4m above the datum. If the pressure at section 1 is 39.24 N/cm^2, find the intensity of pressure at section 2.

Given:

At section 1, $D_1 = 20\text{ cm} = 0.2\text{m}$

$A_1 = \frac{\Pi}{4}(0.2)^2 = 0.314\text{m}^2.$

$P_1 = 39.24\text{ N/cm}^2 = 39.24\text{ X}10^4\text{ N/m}^2.$

$Z_1 = 6.0\text{m}$

At section 2, $D_2 = 0.10\text{m}$

$A_2 = \frac{\Pi}{4}(0.1)^2 = 0.0785\text{m}^2.$

$P_2 = ?$

$Z_1 = 4.0\text{m}$

Rate of flow $Q = 35\text{ lit/sec} = 35/1000 = 0.035\text{m}^3/\text{s}$

$Q = A_1V_1 = A_2V_2$

$V_1 = Q / A_1 = 0.035 / 0.0314 = 1.114\text{ m/s}$

$V_2 = Q / A_2 = 0.035 / 0.0785 = 4.456\text{ m/s.}$

Applying Bernoulli's Equations at sections at 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2$$

Or $(39.24\text{ X}10^4 / 1000\text{X}9.81) + ((1.114)^2/2\text{X}9.81) + 6.0$

$= (p_2/1000X9.81)+((4.456)^2/2X9.81)+4.0$

$40+0.063+6.0 = (p_2/9810)+1.012+4.0$

$46.063 = (p_2/9810) + 5.012$

$(p_2/9810) = 46.063 – 5.012 = 41.051$

$p_2 = (41.051X9810/10^4) = 40.27\ N/cm^2$

PROBLEM -2

In avertical pipe conveying oil of specific gravity 0.8, two pressure gauges have been installed at A and B where the diameters are 16 cm and 8 cm respectively. A is 2 m above B. the pressure gauge readings have shown that the pressure at B is greater than at A by 0.981 N/cm^2. Neglecting all losses, calculate the flow rate. If the gauges at A and B are replaced by tubes filled with the same liquid and connected to a U – tube containing mercury, calculate the difference of level of mercury in the two limbs of the U-tube.

Given:

Sp.gr.. of oil, $S_o = 0.8$

Density, $\varrho = 0.8\ X1000 = 800\ kg/m^3$.

Dia at A, $D_A = 16\ cm = 0.16m$

Area at A, $A_1 = \frac{\Pi}{4} X(0.16)^2 = 0.0201\ m^2$.

Dia. At B $D_B = 8\ cm = 0.08m$

Area at B, $A_B = = \frac{\Pi}{4} X(0.08)^2 = 0.005026\ m^2$

(i). Difference of pressures, $p_B – p_A = 0.981\ N/cm^2$.

$= 0.981\ X\ 10^4\ N/m^2. = 9810\ N/m^2$.

Difference of pressure head $(p_B – p_A)/ \varrho g = (9810 / (800X9.81)) = 1.25$

Applying Bernoulli's theorem at A and B and taking reference line passing through section B, we get

$$\frac{p_A}{\varrho g}+\frac{V_A^2}{2g}+Z_A=\frac{p_B}{\varrho g}+\frac{V_B^2}{2g}+Z_B$$

$$\frac{p_A}{\rho g}-\frac{p_B}{\rho g}+Z_A-Z_B=\frac{V_B^2}{2g}-\frac{V_A^2}{2g}$$

$$\frac{p_A-p_B}{\rho g}+2.0-0.0=\frac{V_B^2}{2g}-\frac{V_A^2}{2g}$$

$$-1.25+2.0=\frac{V_B^2}{2g}-\frac{V_A^2}{2g} \qquad \frac{p_B-p_A}{\rho g}=1.25$$

$$0.75=\frac{V_B^2}{2g}-\frac{V_A^2}{2g} \text{ --------------------- (i)}$$

Now applying continuity equation at A and B, we get

$$V_A X A_1 = V_B X A_2$$

$$V_B=\frac{V_A X A_1}{A_2}=\frac{V_A X \frac{\pi}{4}(.16)^2}{\frac{\pi}{4}(.08)^2}=4V_A$$

Substituting the Value of V_B in equation (i), we get

$$0.75=\frac{16V_A^2}{2g}-\frac{V_A^2}{2g}=\frac{15V_A^2}{2g}$$

$$V_A=\sqrt{\frac{0.75 X 2 X 9.81}{15}}=0.99 m/s.$$

Rate of flow, $Q = V_A X A_1$

$= 0.99 \text{ X } 0.0201 = 0.01989 \text{ m}^3\text{/s}.$

(ii). Difference of mercury in the U –tube.

Let h = difference of mercury level.

Then $h = x\left(\frac{S_g}{S_o}-1\right)$

Where $h=\left(\frac{p_A}{\rho g}+Z_A\right)-\left(\frac{p_B}{\rho g}+Z_B\right)=\frac{p_A-p_B}{\rho g}+Z_A-Z_B$

$= -1.25+2.0-0 = 0.75.$

$$\therefore 0.75 = x\left[\frac{13.6}{0.8} - 1\right] = x \times 16$$

$x = (0.75 / 16) = 0.04687$ cm.

EXPRESSION FOR RATE OF FLOW THROUGH VENTURIMETER.

Venturi meter is a device used for measuring the rate of flow of a fluid flowing through a pipe. It consists of three pats (i). A short converging part (ii) Throat and (iii). Diverging part

Let d_1 = diameter at inlet or at section 1

Let P_1 = pressure at section 1

Let V_1 = velocity of fluid at section 1

Let a_1 = area of section 1 = $\frac{\pi}{4} d_1^2$

And d_2, P_2, V_2, a_2 are the corresponding values at section 2.

Applying the Bernoulli's equation at section 1 & 2

$(P_1/\rho g) + (V_1^2 / 2g) + Z_1 = (P_2/\rho g) + (V_2^2 / 2g) + Z_2$

since the pipe is horizontal $Z_1 = Z_2$

$(P_1/\rho g) + (V_1^2 / 2g) = (P_2/\rho g) + (V_2^2 / 2g)$

$$\frac{P_1 - P_2}{\rho g} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

We know that $\frac{P_1 - P_2}{\rho g}$ is the difference or pressure head and is equal to h.

$$h = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \text{ ------------- (1).}$$

Now applying, continuity equation at 1 & 2

$$a_1V_1 = a_2V_2 \text{ or } V_1 = (a_2V_2/a_1) \text{ --------------- (2).}$$

Sub (2) in equation (1) we get

$$h = \frac{V_2^2}{2g} - \frac{\left(\frac{a_2V_2}{a_1}\right)^2}{2g} = \frac{V_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right]$$

$$V_2^2 = 2gh\,(a_1^2/(a_1^2 - a_2^2))$$

$$V_2 = \sqrt{2gh}.\frac{a_1}{\sqrt{a_1^2 - a_2^2}}$$

Discharge, $Q = a_2V_2$

$$Q = \frac{a_2a_1\sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}} \text{ theoretical discharge}$$

Actual discharge

$$Q_{act} = C_d \text{ X } \frac{a_2a_1\sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

Where C_d = co – efficient of venturi meter.

PROBLEM 3

Water flows through a pipe AB 1.2m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter at C, the pipe branches. Branch CD is 0.8m in diameter and carries one third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.

Solution. Given:

Diameter of Pipe AB, $D_{AB} = 1.2$ m.

Velocity of flow through AB $V_{AB} = 3.0$ m/s.

Dia. of Pipe BC, $D_{BC} = 1.5$m.

Dia. of Branched pipe CD, $D_{CD} = 0.8$m.

Velocity of flow in pipe CE, $V_{CE} = 2.5$ m/s.

Let the rate of flow in pipe AB = Q m^3/s.

Velocity of flow in pipe BC = V_{BC} m^3/s.

Velocity of flow in pipe CD = V_{CD} m^3/s.

Diameter of pipe CE = D_{CE}

Then flow rate through CD = Q / 3

And flow rate through CE = Q – Q/3 = 2Q/3

(i). Now the flow rate through AB = Q = V_{AB} X Area of AB

$= 3 \text{ X } (\pi / 4) \text{ X } (D_{AB})^2 = 3 \text{ X } (\pi / 4) \text{ X } (1.2)^2$

$= 3.393 \text{ m}^3/\text{s}.$

(ii). Applying the continuity equation to pipe AB and pipe BC,

V_{AB} X Area of pipe AB = V_{BC} x Area of Pipe BC

$3 \text{ X } (\pi / 4) \text{ x } (D_{AB})^2 = V_{BC} \text{ x } (\pi / 4) \text{ x } (D_{BC})^2$

$3 \text{ x } (1.2)^2 = V_{BC} \text{ x } (1.5)^2$

$V_{BC} = (3\text{x}1.2^2)/ 1.5^2 = 1.92 \text{ m/s}.$

(iii). The flow rate through pipe

$CD = Q_1 = Q/3 = 3.393 /3 = 1.131 \text{ m}^3/\text{s}.$

$Q_1 = V_{CD}$ x Area of pipe C_D x $(\pi / 4) (C_{CD})^2$

$1.131 = V_{CD} \text{ x } (\pi / 4) \text{ x } (0.8)^2$

$V_{CD} = 1.131 / 0.5026 = 2.25 \text{ m/s}.$

(iv). Flow through CE,

$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262 \text{ m}^3/\text{s}$

$Q_2 = V_{CE}$ X Area of pipe CE = $V_{CE} \text{ X } (\pi / 4) (D_{CE})^2$

$2.263 = 2.5 \text{ X } (\pi / 4) (D_{CE})^2$

$D_{CE} = \sqrt{(2.263 \text{ X}4)/ (2.5 \text{ X } \pi)} = 1.0735 \text{ m}$

Diameter of pipe CE = 1.0735m.

PROBLEM 4

A horizontal Venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.

Given:

$d_1 = 30$ cm

$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (30)^2$

$= 706.85 \text{ cm}^2$

$d_2 = 15$ cm

$a_2 = \frac{\pi}{4} d_2^{\ 2} = \frac{\pi}{4} (15)^2$

$= 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer = x = 20 cm of mercury.

Difference of pressure head, $h = x\left(\frac{S_h}{S_o} - 1\right)$

$= 20\ [\ (13.6 / 1) - 1] = 252.0$ cm of mercury.

$$Q_{act} = C_d \times \frac{a_2 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$= 0.98 \times \frac{706.85 x 176.7 \sqrt{2 x 9.81 x 252}}{\sqrt{706.85^2 - 176.7^2}}$$

$= 125756 \text{ cm}^3 / \text{s}$

= 125.756 lit / s.