Unit 3
Linear Time Invariant Time Systems

Intro:-

The LTI systems are analyzed with the help of:
(i) Impulse response
(ii) Differential eqn (in block dia
(iii) State variable description
(iv) Transfer fn.

Impulse response:

→ Output of the LTI when unit impulse is applied at the input.

Differential Eqn:

→ The impulse response characteristics can be analyzed with the help of constant coefficient differential eqn.

State Variables:

There are two differential eqn basically, these eqns represent LTI system and relates the in to the out.

→ Pictorial visualization of the LTI implemented with the help of scalars, multipliers, adders & integrators.

Transfer fn:

→ The behavior of the LTI can be studied in the frequency domain with the help of transfer fn or LTI fn.

Laplace and Fourier transforms are applied to study LTI fn.
Convolution Integral:

This relates $\delta(t) \ast x(t)$ and unit impulse response of the system.

Representation of $x(t)$ in terms of impulses:

The $\delta(t)$ can be rep as weighted sum of impulses, w.r.t area of impulse from equal to one.

Consider $x(t)$:

| Area under thin pt $A = x(t) \delta(t - \tau) = x(t) \delta(t - \tau)$ |

$A = x(t) \delta(t - \tau)$

(a)

* (a) shows the th

* (a) which is in $x(t)$

* at point $A$ amplitudes

$\tau = \tau$ in $x(t)$

* This pt can be rep as a weighted impulse as in (b). The weight of impulse is $A$.

Since $x(t)$ is continuous, we can represent $x(t)$ in the sum of all of them:

$x(t) = \int_{-\infty}^{\infty} x(t) \delta(t - \tau) d\tau$

Here, integration is used since $x$ is continuous to

The above e.g. rep any CT signal $x(t)$ in terms of weighted impulses.

Derivation of Convolution Integral:

$\delta(t)$ -> 0 if

$y(t)$
Step 1: The c(t) and h(t) can be related as:

\[ y(t) = h(t) \ast x(t) \]

Step 2: Substituting eqn \( x(t) \) in \( y(t) \)

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \]

Step 3: \( x(t) \) is the amplitude of \( x(t) \) at \( t = 2 \). It is constant. So,

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-2) d\tau \rightarrow (2) \]

Step 4: If a unit impulse \( u(t) \) is applied as c(t) to the 6AM, it produces impulse response.

\[ h(t) = u(t) \]

Since the 6AM is LT, \( h(t-2) \) is the response, so \( c(t-2) \) is \( h(t-2) \). 

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \rightarrow (2) \]

Thus, \( y(t) \) is equal to convolution of \( x(t) \) and \( h(t) \).

hence it is called as convolution integral.

Symbolically,

\[ y(t) = x(t) \ast h(t) \]

\[ x(t) \ast h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \]

The \( x(t) \) and \( h(t) \) is given by,

\[ x(t) = e^{-3t} u(t) \]
\[ h(t) = u(t-1) \]

Evaluate the c(t).
The O/p \( y(t) \) = \( \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \).

\[ y(t) = \int_{-\infty}^{\infty} e^{-\pi \tau^2} x(\tau) h(\tau) \, d\tau \]

\( u(t) = 1 \), for \( t \geq 0 \)

\( u(t-2) = 1 \), for \( t-1 \geq 0 \)

\[ y(t) = \int_{-\infty}^{\infty} e^{-\pi \tau^2} x(\tau) h(t-2) \, d\tau \]

\[ y(t) = \int_{-\infty}^{\infty} e^{-\pi \tau^2} x(\tau) h(t-2) \, d\tau \]

\[ y(t) = \frac{1}{\sqrt{8}} \left( e^{-\frac{3}{4}} t^2 \right) \]

This is the required O/p.

**Properties of Convolution:**

1. **Commutative Property:**

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \]

Put \( t-\tau = m \), then \( d\tau = -dm \).

Where \( t = \infty \), \( m = -\infty \), and \( \tau = -\infty \), \( m = \infty \).

\[ y(t) = \int_{-\infty}^{\infty} x(t-m) h(m) \, dm \]

\[ = \int_{-\infty}^{\infty} x(t-m) h(m) \, dm \]

Replace \( m \) by \( \tau \).

\[ y(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) \, d\tau \]
**Associative property:**

Consider the cascade connection

\[ y(t) = y_1(t) \ast h_2(t) \]

\[ y_1(t) = \int_{-\infty}^{\infty} x(t-m) h_1(t-m) \, dm \]

\[ y(t) = \int_{-\infty}^{\infty} y_1(t-m) h_2(t-m) \, dm \]

Put \( t = m + n \), then,

\[ y(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t-m) h_1(t-m-n) \, dn \right] h_2(t-m) \, dm \]

Pulling \( \int \) in \( \int \),

\[ y(t) = \int_{-\infty}^{\infty} x(t-m) h_2(t-n) \, dm \]

So, by,

\[ x(t) * h_1(t) * h_2(t) = x(t) * [h_1(t) * h_2(t)] \]
Distributive Property:

\[ y(t) = x_1(t) + x_2(t) \]

\[ y(t) = \int_{-\infty}^{\infty} x(c) h_1(t-c) + \int_{-\infty}^{\infty} x(c) h_2(t-c) \, dc \]

Impulse response and properties of \( x(t) \):

1. **Dynamically:** (memoryless \( x(t) \)):

   \[ y(t) = \int_{-\infty}^{\infty} x(c) h(t-c) \, dc \]

   By using commutative property,

   \[ y(t) = \int_{-\infty}^{\infty} x(t-c) h(t) \, dc \]

   The \( x(t) \) is said to be static or memoryless if present

   \( y(t) \) depends only on present \( x(t) \).

   Then \( y(t) \) will depend on \( x(t) \) only when \( t=0 \).

   \[ h(t) = 0 \quad \text{for} \quad t \neq 0 \]
The above condition is true only for
\[ h(t) = c \delta(t) \]
\[ y(t) = \int_{-\infty}^{\infty} x(t-T) h(t-T) \, dt \]
\[ = c \int_{-\infty}^{\infty} x(t-T) \delta(t-T) \, dt \]
\[ = c x(t) \delta(t) \]


Well, \( x(t) \delta(t) = x(t) \), hence,
\[ y(t) = c x(t) \]

Thus, the S/H is memoryless or static if
\[ h(t) = c \delta(t) \]

2. **Causality**

Convolution eqn,
\[ y(t) = \int_{-\infty}^{\infty} h(T) x(t-T) \, dT \]

Well, the S/H is said to be causal if its output depends on present and past inputs.
\[ x(t) \rightarrow \text{present input} \]
\[ x(t-T) \rightarrow \text{past input}, \text{for } T \leq 0 \]
\[ x(t+T) \rightarrow \text{future input}, \text{for } T > 0 \]

The result of convolution must be zero for future inputs, i.e., \( T > 0 \), hence the S/H is causal.

For a LTI S/H to be causal,
\[ \text{Causality: } h(t) = 0 \text{ for } t < 0 \]
3. Stability:

\[ y(\tau) = \int_{-\infty}^{\infty} h(\tau) x_1(\tau - \tau) d\tau \]

The magnitude of the op,

\[ |y(\tau)| = \left| \int_{-\infty}^{\infty} h(\tau) x_1(\tau - \tau) d\tau \right| \]

The integrals of eqn,

\[ |y(t)| \leq \int_{-\infty}^{\infty} |h(t)| |x_1(t)| dt \leq \int_{-\infty}^{\infty} |h(t)| dt \]

The LTI system is stable if,

\[ \text{Stability: } \int_{-\infty}^{\infty} |h(t)| dt < \infty \]

4. Step Response:

\[ y(t) = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) d\tau \]

Let, \( u(t) = u(t) = 1 \) for \( t \geq 0 \).

\[ x(t) = u(t) u(t - 2) = 1 \text{ for } t \geq 0 \]

The above eqn can be written as,

\[ x(t) = u(t - 2) = 1 \text{ for } t \geq 2 \]

Thus the step response becomes,

\[ y(t) = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) d\tau \]

\[ \text{Step Response: } = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) d\tau \]
Differential equations:

The general form of a constant coefficient differential eqn is,

\[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \] \( \text{ Eqn 1) } \)

\( N \) - order of differential eqn, \( x(t) \), \( y(t) \) is output.

The solution has two components:

(i) Natural response
(ii) Forced response.
(iii) Complete response (equal to sum of above two)

**Natural response:** \( y_n(t) \):

Initial condition = 2 \( \text{no}. \)

hence eqn (1) becomes,

\[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \] \( \text{ Eqn 2) } \)

This is a homogeneous differential equation. The natural response of the system is,

\[ y_n(t) = \sum_{i=1}^{r} c_i e^{r_i t} \]

\( r_i \) - real and distinct \( L \in \mathbb{R} \)

**Roots of \( N \):**

\[ \sum_{k=0}^{N} a_k x^k = 0 \]

roots \( c_i \) of above eqn.

If \( r_i \) is an \( \text{real number} \) then \( y_n(t) = e^{r_i t} (c_{i, \text{even}} + c_{i, \text{odd}} \sin(t) + c_{i, \text{odd}} \cos(t)) \)

If \( r_i \) is repeated, \( i.e. r_i+1 \) then,
Determine the natural response of the system.

\[
\frac{dy(t)}{dt} + ay(t) = x(t) \quad \text{with} \quad y(0) = 0.
\]

**Solution:**

Assume \( x(t) = 0 \).

Then, \( \frac{dy(t)}{dt} + ay(t) = 0 \).

The characteristic equation is:

\[
\sum a_k r^k = 0.
\]

For \( a = 1, a_1 = -a \),

\[
8 + 10r_1 = 0.
\]

Solving for \( r_1 \):

\[
r_1 = -\frac{10}{10} = -0.2.
\]

The natural response is:

\[
y_n(t) = c_1 e^{-0.2t}.
\]

Substitute \( r_1 \) in the above eqn.

\[
y(t) = c_1 e^{-0.2t}.
\]

\( c_1 \) is determined by initial condition \( y(0) = 0 \), put \( t = 0 \).

\[
y(0) = c_1,
\]

\[
0 = c_1 \quad \text{since} \quad y(0) = y_n(0) = 0.
\]

Hence,

\[
y_n(t) = 0 e^{-0.2t}.
\]

This is the natural response of the given system.
2. **Forced response** \( y_f(t) \):

- **Initial condition** = 0.
- It has two components:
  1. The term similar to natural response
  2. A particular solution.

Typical form and \( e^{at} \) terms of particular soln.

<table>
<thead>
<tr>
<th>No.</th>
<th>( e^{at} )</th>
<th>Particular soln ( y_f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( k )</td>
</tr>
<tr>
<td>2</td>
<td>( e^{-at} )</td>
<td>( ke^{-at} )</td>
</tr>
<tr>
<td>3</td>
<td>( \cos(wt+\phi) )</td>
<td>( k_1 \cos(wt) + k_2 \sin(wt) )</td>
</tr>
<tr>
<td>4</td>
<td>( \cos(wt+\phi) )</td>
<td>( e^{-at}(k_1 \cos(wt) + k_2 \sin(wt)) )</td>
</tr>
<tr>
<td>5</td>
<td>( t )</td>
<td>( ko + k_1 t )</td>
</tr>
<tr>
<td>6</td>
<td>( t^p )</td>
<td>( ko + k_1 t + k_2 t^2 + \ldots + k_p t^p )</td>
</tr>
<tr>
<td>7</td>
<td>( te^{-at} )</td>
<td>( e^{-at}(ko + k_1 t) )</td>
</tr>
</tbody>
</table>

Ex3: Determine the forced response.

\[
\frac{dy(t)}{dt} + 10 y(t) = 2 x(t) \quad x(t) = 2 u(t)
\]

\[
y(t) = y_n(t) + y_f(t)
\]

**NPI: characteristic eqm**

For the order \( n = 1 \),

\[
\sum a_k x^k = 0
\]

i.e., \( a_0 + a_1 t + \ldots + a_n t^n = 0 \)

\[
(10 + 5) t^0 \neq 0
\]
Steps:

1. To obtain natural response,
   \[ n = -a, \ \alpha \text{ real.} \]
   \[ y(t) = c_1 e^{\alpha t} \]
   \[ y(0) = c_1 e^{-2t} \]

2. Particular solution,
   \[ x(t) = \text{a constant} \]
   \[ y_p(t) = k \]

3. The values of constants in \( y_p(t) \).
   \[ 5 \frac{d}{dt}(k) + 10(k) = 8 \times 4u(t) \]
   \[ 0 + 10k = 4 \]
   \[ k = \frac{4}{10} = 0.4 \]
   
   Hence, \( y(0) = 0.4 \).

4. To determine, \( y(t) = y_n(t) + y_p(t) \)
   \[ y(t) = c_1 e^{\alpha t} + 0.4 \]

5. Obtain constants of \( y_n(t) \) with initial condition
   \[ y(0) = 0 \]
   \[ c_1 = c_1 e^{-2t} + 0.4 \]
   \[ c_1 = 0.4 \]
   
   Hence, \( y(t) = -0.4 e^{-2t} + 0.4 \)

\[ y(t) = 0.4 (1 - e^{-2t}) \]
Block diagram representation:
Blocks that are used for block diagram rep.

1. Scalar multiplication:
   \[ x(t) \rightarrow a \cdot y(t) = a \cdot x(t) \]
   If \( x(t) \) is multiplied by a constant \( a \), \( y(t) = a \cdot x(t) \)

2. Addition:
   \[ x(t) \quad \oplus \quad y(t) = x(t) + y(t) \]
   \( \oplus \) addition operation

3. Integration:
   \[ \int_{-\infty}^{t} x(\tau) d\tau \rightarrow y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

WKT CT/TM are described by diff. eqns. These diff.
eqns are converted into an integration. Because the
implementation of integrator is easier than differentiator.
Integrator reduces noise and differentiator amplifies
noise.

\[ \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) - \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \rightarrow 0 \]

The integration operation can be rep as,
\[ y^{(n)}(t) = \int_{-\infty}^{t} y^{(n-1)}(\tau) d\tau \]

Here, \( y^{(n)}(t) \) is \( n \)-fold integration of \( y(t) \)
with time.

\[ \frac{d}{dt} y^{(n)}(t) = y^{(n-1)}(t) \]

If we integrate \( n \) times, we get,
\[ a_0 y^{(2)}(t) + a_1 y^{(1)}(t) + a_2 y(t) = b_0 x^{(2)}(t) + b_1 x^{(1)}(t) + b_2 x(t) \]

Normally, \( a_2 = 1 \), then above eqn.

\[ y(t) = -a_1 y^{(1)}(t) - a_0 y^{(0)}(t) + b_2 x(t) + b_1 x^{(1)}(t) + \]

**Direct form-I implementation**

The direct form-I implementation of eqn. 8 is below.

**Direct form-II implementation**

Two SIs are cascaded, one SI \( \rightarrow \) present past input, one SI \( \rightarrow \) past output.
Now the cascade shown in above figure is as given

\[ x(t) \rightarrow \text{M1} \rightarrow \text{M2} \rightarrow y(t) \]

\[ x(t) \rightarrow \text{M1} \rightarrow \text{M2} \rightarrow y(t) \]

Cascade of \( \text{M1} \) and \( \text{M2} \) can be interchanged. Note cascading sequence does not affect the operation of the \( \text{Mm} \).

\[
\begin{align*}
& x(t) \\
& + \\
& \text{+} \\
& 2(t) \\
& + \\
& 2(2(t)) \\
& \downarrow \text{b1} \\
& \downarrow \text{b2} \\
& y(t)
\end{align*}
\]

2(t) is integrated by two integrators and both produce \( 2(t) \). Hence only one integrator can be used.

\[
\begin{align*}
& x(t) \\
& + \\
& \text{+} \\
& 2(t) \\
& + \\
& 2(2(t)) \\
& \downarrow \text{b1} \\
& \downarrow \text{b2} \\
& y(t)
\end{align*}
\]

Direct form-II Implementation.
(Ex.) Draw the direct form-I and II implementation of the system described by the following.

\[
\frac{dy(t)}{dt} + 5y(t) = 3x(t)
\]

\[y(0) = 1, \text{ hence,}
\]

\[y(t) + 5y(t-1) = 3x(t) + 3x(t-1)
\]

\[y(t) = -5y(t-1) + 3x(t-1)
\]

and \( a_1 = 5, \text{ and } b_1 = 8 \)

Direct form-I

[Diagram of Direct form-I]

Direct form-II

[Diagram of Direct form-II]

Fourier Methods for Analysis:

Frequency Response:

The convolution is,

\[y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \, d\tau
\]

Let the impulse be \(e^{j\omega t}\) \(\text{ie, sinusoid}\)

Then it becomes,

\[y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)} \, d\tau
\]
In the integral representation, Fourier transform of \( h(t) \)

\[ \text{ie, } y(t) = e^{j\omega t} H(j\omega) \]

\( H(j\omega) \) is the frequency response of \( h(t) \).

Again consider the convolution,

\[ y(t) = x(t) * h(t) \]

By convolution property of \( F.T \), we can write,

\[ Y(j\omega) = X(j\omega) H(j\omega) \]

\[ y(t) = \frac{Y(j\omega)}{X(j\omega)} H(t) \]

or \( Y(j\omega) = X(j\omega) H(j\omega) \)

(Ex.): \( x(t) = u(t) \) is given as,

\[ h(t) = \frac{1}{Rc} e^{-\frac{t}{Rc}} u(t) \]

Determine the impulse response, freq response and plot the magnitude phase response.

\[ H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt \]

\[ = \frac{1}{Rc} \int_{0}^{\infty} e^{-\frac{t}{Rc}} e^{-j\omega t} \, dt \]

\[ = \frac{1}{Rc} \int_{0}^{\infty} e^{-t\left(\frac{\omega}{Rc} + 1\right)} \, dt \]

\[ = \frac{1}{Rc} e^{-\frac{\omega}{Rc}} \int_{0}^{\infty} e^{-t} \, dt \]
\[ H(\omega) = \frac{1}{Re} \left[ 1 - \frac{j\omega R_C}{\omega} \right] \int e^{-t} \left( j\omega + \frac{1}{R_C} \right) dt \]

To determine mag & phase response, let us rearrange the eqn as,

\[ H(\omega) = \frac{1}{1 + j\omega R_C} \frac{1 - j\omega R_C}{1 + j\omega R_C} \]

\[ = \frac{1}{1 + (j\omega R_C)^2} + \frac{j\omega R_C}{2} \]

\[ |H(\omega)| = \sqrt{\frac{1}{1 + (j\omega R_C)^2} + \frac{(j\omega R_C)^2}{1 + (j\omega R_C)^2}} \]

\[ |H(\omega)| = \frac{1}{\sqrt{1 + (j\omega R_C)^2}} \]

Phase response, \( \angle H(\omega) = \tan^{-1} \left( \frac{-j\omega R_C}{1 + (j\omega R_C)^2} \right) \]

\[ = \tan^{-1}(\omega R_C) \]

Let \( R_C = 1 \), the mag & phase response will be,

\[ |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \]

\[ \angle H(\omega) = -\tan^{-1}(\omega) \]
Differential equations

This rep is useful in obtaining freq response and
impulse response of the sim.
Consider the diff. eqn.

\[ \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \]

The differentiation property of \( \mathcal{F} \) is,

\[ \frac{d}{dt} x(t) \xrightarrow{\mathcal{F}} j\omega x(j\omega) \]

\[ \Rightarrow \frac{d}{dt} x(t) \xrightarrow{\mathcal{F}} j\omega x(j\omega) \]

Let us apply this property.

\[ \sum_{k=0}^{N} a_k (j\omega)^k y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k x(j\omega) \]

\[ \frac{y(j\omega)}{x(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \]

This is the sim transfer fn,

\( H(j\omega) = \frac{y(j\omega)}{x(j\omega)} \), The impulse response & freq response
\( x(j\omega) \) can be obtained.

Ex. 1:
The diff. eqn of the sim,

\[ \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + by(t) = -dx(t) \]

Determine freq & impulse response.

By eqn (1), the TF is,

\[ \mathcal{F} \{ y(t) \} + 5 \mathcal{F} \{ y(t) \} + b \mathcal{F} \{ x(t) \} = -d \mathcal{F} \{ x(t) \} \]
\[ Y(j\omega) = \frac{1}{(j\omega)^2 + 5j\omega + 6} = -j\omega x(j\omega) \]

\[ H(j\omega) = \frac{\frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}}{j\omega^2 + 5j\omega + 6} \]

**Freq response**

\[ H(j\omega) = \frac{-j\omega}{j\omega^2 + 5j\omega + 6} \]

(by partial fraction)

\[ = \frac{\frac{2}{j\omega + 2} - \frac{1}{j\omega + 3}}{j\omega^2 + 5j\omega + 6} \]

With

\[ e^{-at}u(t) \xrightarrow{L} \frac{1}{a+j\omega} \]

\[ h(t) = \frac{\Gamma(t) \{ H(j\omega) \}}{\Gamma(j\omega)} \]

\[ = (2e^{-2t} - 3e^{-3t})u(t) \]

This is the impulse response of the LTI.

**Laplace Transform Analysis**

1. **Eigenv and Eigenvalues:**

   The O/P of LTI - CT S/I, u,
   \[ y(t) = \int_{-\infty}^{\infty} h(t) x(t-\tau) d\tau \]

   Let the LTI S/I be excited by the complex exponential \( s(t) \).

   \[ e^{s} = e^{(a+j\omega)t} = e^{at} \cdot e^{j\omega t} \]
\[ e^{at}(\cos \omega t + j \sin \omega t) \]
\[ = e^{at}(\cos \omega t) + e^{at}j \sin \omega t \]

When \( x(t) = e^{at} \)

Then \( y(t) = \int_{-\infty}^{\infty} h(t) e^{a(t-t')} dt' \)

\[ = \int_{-\infty}^{\infty} h(t) e^{at} e^{-a t} dt' \]
\[ = e^{at} \int_{-\infty}^{\infty} h(t) e^{-a t} dt' \]

\[ H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \]

\( H(s) \) is called a transfer function theorem.

\[ y(t) = H(s) e^{at} \]

\( e^{at} \) is an input, \( s \) is \( H(s) \) is an output.

Solution of differential equations :

The differentiation theorem is used.

\[ L\left( \frac{dy(t)}{dt} + 5y(t) \right) = s \cdot y(s) - y(0^-) \]

\[ (s^2) : \]

\[ \frac{dy(t)}{dt} + 5y(t) = x(t) \]

With initial condition \( y(0^-) = 0 \), \( \dot{y}(0^-) = 1 \)

\[ (s^2) : \]

\[ \frac{dy(t)}{dt} + 5y(t) = x(t) \]

Take Laplace transform,

\[ sY(s) - y(0^-) + 5Y(s) = X(s) \]

Initial condition \( y(0^-) = y(0^-) = 0 \)
\( x(t) = 3e^{-2t} \cdot u(t) \)

\[ e^t \left\{ \frac{1}{s-a} \right\} = \frac{1}{s-a} \]

\[ x(s) = \left\{ \frac{3e^{-2t} \cdot u(t)}{s^2 + 2} \right\} = 3 \cdot \frac{1}{s^2 + 2} \]

Put \( y(0^-) \) and \( x(\infty) \) in the eqn.

\[ s \cdot y(s) + 2 + 5y(s) = \frac{3}{s^2 + 2} \]

\[ y(s) + 2 + 5 \cdot \frac{3}{s^2 + 2} = \frac{3}{s^2 + 2} \]

\[ y(s) = \frac{3}{s^2 + 2} - 2 \cdot \frac{3}{s^2 + 2} \]

\[ y(s) = \frac{3}{s^2 + 5} \]

By using partial fraction,

\[ y(s) = \frac{1}{s^2 + 5} - \frac{3}{s^4 + 5} \]

Using inverse L.T.,

\[ y(t) = e^{-2t} \cdot u(t) - 3e^{-5t} \cdot u(t) \]

System Transfer fn.

Write the eqn of LTI - CT

\[ y(t) = h(t) = x(t) \]

Taking L.T.,

\[ Y(s) = H(s) \cdot X(s) \]

\[ \frac{H(s)}{X(s)} = Y(s) \]

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The impulse response can be obtained by taking inverse L.T of \( H(s) \).

The c.m. transfer in cann also be obtained from the differential eqn.

\[
\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)
\]

\( y(t) = e^{st} \) then \( y(t) = H(s) e^{st} \)

put this values in the above eqn.

\[
\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} H(s) e^{st} = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} e^{st}
\]

\( \frac{d^k}{dt^k} e^{st} = s^k e^{st} \) then we get

\[
H(s) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} s^k
\]

Inverse systems:

\[
\begin{align*}
\text{Input} & \quad \rightarrow \quad h(t) \quad \rightarrow \quad x(t) \\
\text{Inversion} & \quad \rightarrow \quad s \cdot h(t) \quad \rightarrow \quad x(t)
\end{align*}
\]

Consider the \( s \cdot h(t) \) connected in cascade as

The i/p \( x(t) \) is given to the cascade connection.

Then \( y(t) \) produced is same as input i.e., \( x(t) \).

Thus \( s \cdot h(t) \) satisfies the condition.

\[
h(t) + h(t) = 2s(t)
\]
State equations and matrix:

State Variable Description:
The state is the minimal set of signals that represent the system's entire past memory.

The state equations for the LTI system are written as:

\[
\begin{align*}
\frac{d}{dt} q(t) &= A q(t) + b x(t) \\
q(t) &= C q(t) + D x(t)
\end{align*}
\]

Here A, b, c, d are the matrices representing internal structure of the system. The memory of the system is contained in inductors and capacitors. Hence, voltage across capacitor or current through an inductor can be considered as state of the system.

Example:

\[
\begin{align*}
\frac{d}{dt} q_1(t) &= q_2(t) + x(t) \\
\frac{d}{dt} q_2(t) &= q_1(t) + x(t)
\end{align*}
\]

The states, \( q_1(t) \) and \( q_2(t) \), are the output of two integrators. Hence previous states are \( \frac{dq_1(t)}{dt} \) and \( \frac{dq_2(t)}{dt} \).

From figure,

\[
\frac{dq_1(t)}{dt} = q_2(t) - q_1(t) - 2q_1(t) + x(t)
\]
\[
\frac{dq_1}{dt} = q_1(t) 
\]
\[
y(t) = 3q_1 + q_2(t) + 2 \cdot x(t) 
\]

Eqs. 0 and 0 can be expressed in the matrix form,

\[
\begin{bmatrix}
\frac{dq_1(t)}{dt} \\
\frac{dq_2(t)}{dt}
\end{bmatrix} =
\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}
\begin{bmatrix}
q_1(t) \\
q_2(t)
\end{bmatrix} +
\begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t)
\]

Here, let,

\[
\frac{dq(t)}{dt} = \begin{bmatrix}
\frac{dq_1(t)}{dt} \\
\frac{dq_2(t)}{dt}
\end{bmatrix}, \quad q(t) = \begin{bmatrix} q_1(t) \\
q_2(t)\end{bmatrix}
\]

Eqn (5) becomes,

\[
\frac{dq(t)}{dt} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t)
\]

Compare with the standard form,

\[
A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Similarly, Eqn 6 can be expressed as,

\[
y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\
q_2(t)\end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} x(t)
\]

\[
y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} q(t) + \begin{bmatrix} 2 \end{bmatrix} x(t)
\]

On comparing,

\[
c = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \end{bmatrix}
\]

Eqs. 0 and 6 are collectively called as

state variable description of the given
Frequency response using state variables

Consider, \( \frac{dq(t)}{dt} = Aq(t) + b x(t) \)

take LFT of the above eqn.

\[ jw \quad q(jw) = A \quad q(jw) + b \quad x(jw) \]

\[ q(jw) = \begin{bmatrix} q_1(jw) \\ \vdots \\ q_n(jw) \end{bmatrix} \]

\[ b \quad x(jw) = \begin{bmatrix} b_1(jw) \\ \vdots \\ b_n(jw) \end{bmatrix} \]

We can write eqn 1 as:

\[ jw \quad q(jw) - A \quad q(jw) = b \quad x(jw) \]

\[ (jwI - A) \quad q(jw) = b \quad x(jw) \]

I is identity matrix

\[ \quad q(jw) = (jwI - A)^{-1} \quad b \quad x(jw) \]

Take I.F of \( y(t) = c \quad q(t) + d \quad x(t) \)

\[ Y(jw) = c \quad q(jw) + d \quad X(jw) \]

Sub 1 in 2,

\[ Y(jw) = c \left( (jwI - A)^{-1} \quad b \quad X(jw) \right) + d \quad X(jw) \]

\[ Y(jw) = \left[ c \left( (jwI - A)^{-1} \quad b + d \right) \right] \quad X(jw) \]

\[ H(jw) = c \left( (jwI - A)^{-1} \quad b + d \right) \]

This eqn gives frequency response of the SSM.
Transfer fn of state variables:

Consider,
\[ \frac{dq(t)}{dt} = Aq(t) + bx(t) \]  \( \rightarrow (1) \)

Take L.T of the eqn \( (1) \)

\[ SQ(s) = AQ(s) + bx(s) \]  \( \rightarrow (2) \)

Here, \( q(s) = \begin{bmatrix} q_1(s) \\ q_2(s) \\ \vdots \\ q_n(s) \end{bmatrix} \)

We can write eqn \( (1) \) as,

\[ SQ(s) - AQ(s) = bx(s) \]

\[ (S - A) q(s) = bx(s) \]

\[ q(s) = (S - A)^{-1} bx(s) \]  \( \rightarrow (3) \)

Consider

\[ y(t) = c q(t) + dx(t) \]  \( \rightarrow (4) \)

Take L.T of \( (4) \),

\[ Y(s) = Cq(s) + Dx(s) \]  \( \rightarrow (5) \)

Put eqn \( (3) \) in \( (5) \),

\[ Y(s) = C (S - A)^{-1} bx(s) + D x(s) \]

\[ Y(s) = \left\{ (S - A)^{-1} b + D \right\} x(s) \]

\[ \frac{Y(s)}{X(s)} = \left\{ (S - A)^{-1} b + D \right\} \]  \( \rightarrow \) Transfer fn.
Determine the transfer fn of the cm discussed earlier.

\[ A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 8 & 1 \end{bmatrix}, \quad D = C \begin{bmatrix} 2 \end{bmatrix}. \]

\[ (8I - A)^{-1} = \begin{bmatrix} 3 - 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{s^2 - 2s + 1} \begin{bmatrix} 3 & -1 \\ 1 & 5 - 2 \end{bmatrix}. \]

\[ H(s) = C (8I - A)^{-1} b + D. \]

\[ H(s) = \begin{bmatrix} 8 & 1 \end{bmatrix} \frac{1}{s^2 - 2s + 1} \begin{bmatrix} s & -1 \\ 1 & s - 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \]

\[ = \frac{1}{s^2 - 2s + 1} \begin{bmatrix} 8 & 1 \\ 1 & s - 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2. \]

\[ = \frac{1}{s^2 - 2s + 1} \begin{bmatrix} 8s + 1 & s - 5 \\ 1 & s - 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2. \]

\[ H(s) = \frac{8s^2 - 5 + S}{s^2 - 2s + 1}. \]

Transfer fn.