Deflection of Beams

If a beam carries uniformly distributed load or a point load, the beam is deflected from its original position.

In this chapter, we're going to study the amount by which a beam is deflected from its position.

Deflection & Slope of a Beam Subjected to Uniform Bending Moment

Let:

- \( R \) → Radius of curvature of the deflected beam.
- \( y \) → deflection of beam at the centre.
- \( I \) → Moment of Inertia of beam section.
- \( E \) → Young's modulus for the beam material.
- \( \theta \) → slope of the beam at the end A.
Hence \( \tan \theta = 0 \). When \( \theta \) is in radians,

\[
\frac{dy}{dx} = \tan \theta = 0
\]

\( Ac = Bc = \frac{1}{2} \)

\( Ac \times cB = Dc \times cc' \)

\( \frac{1}{2} \times \frac{1}{2} = (2R-y) \times y \)

\[
\frac{L^2}{4} = 2Ry - y^2
\]

\[
\frac{L^2}{4} = 2Ry
\]

\[ y = \frac{L^2}{8} R \]  

Bending Moment equation,

\[ \frac{M}{J} = \frac{F}{R} \Rightarrow R = \frac{F \times L}{M} \]

\[ y = \frac{L^2}{8 \times E} \frac{F}{M} \Rightarrow y = \frac{M^2}{8EJ} \]  

Deflexion

\[ \sin \theta = \frac{1}{2R} \]

angle \( \theta \) is very small: \( \sin \theta = \theta \)

\[ \theta = \frac{1}{2R} \]

\[ \theta = \frac{1}{2R} \]

\[ \theta = \frac{M \times L}{2EJ} \]
Relation between slope, deflection & radius of curvatures:

Deflection = y

Slope = \frac{dy}{dx}

Bending moment = EI \frac{d^2y}{dx^2}

Shearing force = EI \frac{d^3y}{dx^3}

The rate of loading = EI \frac{d^4y}{dx^4}

Units: In the above eqns. E is taken as N/mm², I is taken in mm⁴, y is taken in mm, M is taken in Nm & x is taken in m.

Methods of determining slope & deflection at a section in a loaded beam:

The following are the important methods for finding the slope & deflection at a section in a loaded beam:

i) Double integration method.

ii) Moment Area method.

iii) Macaulay’s method.
In case of double integration method,

\[ M = E I \frac{d^2 y}{dx^2} \text{ (or) } \frac{d^2 y}{dx^2} = \frac{M}{E I} \]

First integration of the above eqn gives

Value of \( \frac{dy}{dx} \) (or, slope). The second integration gives the value of vertical deflection.

The first two methods are used for a static load whereas the third method is used for moving load.

Deflection of SSSB carrying a point load at the end.

\[ R_A = R_B = \frac{w}{2} \]

Consider a section \( x \) at a distance \( x \) from \( A \). The bending moment at this section is given by

\[ M_x = R_A \times x \]

\[ = \frac{w}{2} \times x \times x \]
\[ M = E I \frac{d^2 y}{dx^2} \]

\[ E I \frac{d^2 y}{dx^2} = \frac{W}{2} \times x. \]

On integration, we get:

\[ \int E I \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1. \]

Where \( C_1 \) is the constant of integration.

The boundary condition is that at \( x = \frac{L}{2} \),

\[ \text{Slope} \left( \frac{dy}{dx} \right) = 0 \] \[ \text{As the max. deflection is at the centre, hence slope at the centre will be zero}. \]

Subs. this boundary condition into eqn. we get:

\[ 0 = \frac{W}{4} \times \left( \frac{L}{2} \right)^2 + C_1. \]

\[ C_1 = -\frac{Wl^2}{16}. \]

Substituting the value of \( C_1 \) in eqn.

\[ E I \frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}. \]

The above eqn. is slope eqn.

Slope is maximum at \( A \). At \( A \) \[ x = 0 \] hence slope at \( A \) will be:

\[ \int \left( \frac{dy}{dx} \right)_A \, dx = \frac{W}{4} \times 0 - \frac{Wl^2}{16}. \]

\( \left( \frac{dy}{dx} \right)_A \) is represented by \( C_1 \).
\[ \sum I = 0 \Rightarrow - \frac{wL^2}{16} \]

\[ \theta_A = - \frac{wL^2}{16EI} \quad \Rightarrow \quad \theta_A = \theta_B \]

\[ \theta_A = \theta_B = - \frac{wL^2}{16EI} \]

It gives the slope in radians.

Deflection at any point:

\[ EI \times y = \frac{W}{J_1} \cdot \frac{x^3}{3} - \frac{wL^2}{16} \cdot x + c_2 \]

\[ c_2 \quad \text{a constant of integration} \]

\[ EI \times y = 0 - 0 + c_2 \quad (\text{At } x = 0) \]

\[ c_2 = 0 \]

\[ EI \times y = \frac{Wx^3}{12} - \frac{wL^2}{16} \cdot x \]

Where \( x = \frac{L}{2} \):

\[ EI \times y_e = \frac{W}{12} \left( \frac{L}{2} \right)^3 - \frac{wL^2}{16} \frac{L}{2} \]

\[ = \frac{W L^3}{96} - \frac{wL^3}{32} \]

\[ y_e = - \frac{wL^3}{48EI} \]

Note: This equation is the deflection at any point under the given conditions.
Deflection of a SS8 with an eccentric Pt. Lc.

\[ \theta_A = -\frac{W \cdot a \cdot b}{6EI} \]

\[ \gamma_c = \frac{W a^2 \cdot b^2}{3EI} \]

\[ \gamma_{\text{max}} = \frac{W \cdot b}{9\sqrt{3} \cdot EI} \left( a^2 + 2ab \right)^{3/2} \]

Ph.2

Determine the slope at left support, deflection under the load & max. deflection of a SS8 of length 5m, which is carrying a Pt. load of 5kN at a distance 3m from the left end. Take \( E = 2 \times 10^5 \) N/mm² and \( I = 1 \times 10^8 \) mm⁴.

Solution:

\[ 5 \text{kN} \]
Length $L = 5 \text{m} = 5000 \text{mm}$

$W = 5 \text{kN} = 5 \times 10^3 \text{N}$

$a = 3 \text{m} = 3000 \text{mm}$

$b = L - a = 5 - 3 = 2 \text{m} = 2000 \text{mm}$

$I = 2 \times 10^5 \text{N/mm}^2$

$I = 1 \times 10^8 \text{mm}^4$

$\theta_a = \frac{-W \cdot a \cdot b}{6 \cdot E \cdot I \cdot L}$

$= \frac{-5000 \times 3000 \times 2000}{6 \times 2 \times 10^5 \times 10^2 \times 5000} \times \frac{(3000 + 2000)}{2 \times 2000}$

$\theta_a = -0.00035 \text{ radians}$

$y_c = \frac{W \cdot a^2 \cdot b^2}{3 \cdot E \cdot I}$

$= \frac{5000 \times 3000^2 \times 2000^2}{3 \times 2 \times 10^5 \times 10^2 \times 5000}$

$= 0.6 \text{mm}$

$y_{\text{max}} = \frac{W \cdot b}{9 \sqrt{3} \cdot E \cdot I \cdot L}$

$= \frac{5000 \times 2000 \left(3000^2 + 2 \times 2000 \times 3000 \right)^{3/2}}{9 \times \sqrt{3} \times 2 \times 10^5 \times 10^2 \times 5000}$
A beam of length 3m and of uniform rectangular cross-section 80 x 100 mm at its ends. It carries a uniformly distributed load of 9 kN/m over the entire length. Determine the width & depth of the beam if maximum permissible bending stress is 75 N/mm² & centre deflection not to exceed 10 mm. E = 1 x 10⁴ N/mm² 

\[ F_a = \frac{Wl^2}{24EI} \]

\[ \sigma = \frac{M}{W} \text{ (for bending)} \]

\[ \delta = \frac{5WL^3}{384EI} \]

Load: \[ W = 9 \times 10 = 90 kN \]

\[ W = 9 \times 10 = 90 kN \]

Contact load: \[ W = 9 \times 5 = 45 kN \]

\[ W = 9 \times 5 = 45 kN \]

\[ E = 1 \times 10^4 N/mm^2 \]

\[ I = \frac{bh^3}{12} \]

\[ I = \frac{80 \times 100^3}{12} = 6.67 \times 10^8 mm^4 \]

\[ M = \frac{90 \times 80}{2} = 3600 kN.mm \]

\[ \sigma = \frac{3600}{6.67 \times 10^8} \times 10^4 = 0.53 \text{ kN/mm}^2 \]

\[ \sigma = \frac{45 \times 80}{6.67 \times 10^8} \times 10^4 = 0.36 \text{ kN/mm}^2 \]

\[ \delta = \frac{5 \times 90 \times 80^3}{384 \times 1 \times 10^4 \times 6.67 \times 10^8} = 0.00068 mm \]

\[ \delta = \frac{5 \times 45 \times 80^3}{384 \times 1 \times 10^4 \times 6.67 \times 10^8} = 0.00041 mm \]
\[ l d^2 = \frac{28125000 \times 12}{14} = 24107142.85 \text{ mm}^2 \]

\[ a = \frac{8.38706 \times 10^7}{24107142.85} = 364.58 \text{ mm} \]

\[ b \times (364.58)^2 = 24107142.85 \]

\[ b = 181.36 \text{ mm} \]

**Macauley's Method:**

This method was devised by Mr. M. H. Macaulay & is known as Macauley's method.

This method mainly consists in the special manner in which the bending moment at any section is expressed & in the manner in which the integrations are carried out.

**Deflection of a SS8 with an Eccentric Point load:**

![Diagram of a SS8 with an eccentric point load]
\[ R_k = \frac{W \cdot b}{L} \quad \text{so} \quad R_k = \frac{2W \cdot a}{L} \]

The bending moment at any section A B C D at a distance \( x \) from A is given by:

\[ M_n = R_A \cdot x = \frac{W \cdot b \cdot x}{L} \]

\[ M_n = R_A \cdot x = W \cdot x \left( x - a \right) \]

\[ M_n = \frac{W \cdot b \cdot x}{L} = \frac{W \cdot (x - a)}{L} \]

\[ M = \int \int d^2 y / dx^2 \]

\[ \int d^2 y / dx^2 = \frac{W \cdot b \cdot x}{L} \left( x - a \right) \]

\[ \int dy / dx = \frac{W \cdot b \cdot x^2}{L} + c_1 \]

\[ \frac{(x - a)^2}{2} \quad \text{and not} \quad \frac{x^2}{2} = ax \]

Integrating eqn.

\[ \int I y = \frac{W \cdot b \cdot x^3}{2} + (c_1 + c_2) \left( -\frac{W (x - a)^3}{2} \right) \]

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\[ \Delta A = - \frac{Wb}{6EI} \left( 1^2 - l^2 \right) \]

\[ Y_c = - \frac{wa^2 l^2}{3EI} \]

**Prob.** A beam of length 6m is simply supported at ends and carries two point loads of 48 kN & 40 kN at a distance of 1m & 3m respectively from the left support. Find (i) Deflection under each load.

(ii) Max. deflection and

(iii) The point at which max. deflection occurs.

Take \( E = 2 \times 10^5 \text{ N/mm}^2 \) & \( J = 8.5 \times 10^5 \text{ mm}^4 \)

**Soh:**

![Diagram of a simply supported beam with loads and reactions](image)

**Given:**

\[ J = 8.5 \times 10^5 \text{ mm}^4 \]

\[ R_B x 6 = 48 x 1 + 40 x 3 \]

\[ R_B = \frac{168}{6} = 28 \text{ kN} \]

\[ R_A = \text{Total load} - R_B = (48 + 40) \text{ kN} \]

\[ \frac{EJ}{d^2y}{dx^2} = R_A x \]

\[ -48(x - 1), -40(x^2) \]

\[ 60x \]

\[ -48(x - 1), -40(x^2) \]
Integrating above equation.

\[ \int \frac{dy}{dx} = \left( \frac{6x^2}{2} + c_1 \right) \left( -\frac{48(x-1)^2}{2} \right) \left( -\frac{40(x-3)^2}{2} \right) \]

\[ = 36x^2 + c_1 \left( -24(x-1)^2 \right) \left( -20(x-3)^2 \right). \]

\[ \int 2y = 10x^3 + c_1(x+2) \left( -8(x-1)^3 \right) \left( -\frac{20}{3}(x-3)^3 \right). \]

\[ C_2 = 0. \]

(iii) at \( x = 6 \) m: \( y = 0. \)

\[ 0 = 10 \times 6^3 + c_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3. \]

\[ 0 = 2160 + 6c_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3. \]

\[ C_1 = -163.333. \]

\[ \int 2y = 10x^3 - 163.333x \left( -8(x-1)^3 \right) - \frac{20}{3}(x-3)^3. \]

\[ \int 2 \cdot y_c = 10 \times 1^3 - 163.333 \times 1. \]

\[ = -153.333 \text{ kN m}^3. \]

\[ y_c = \frac{-153.333 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}. \]

\[ = -9.019 \text{ mm}. \]

The negative sign indicates load downward.
Deflection under Second Load

\[ \begin{align*}
F_2 : y_2 &= 10 \times 3^3 - 163.33 \times 3 - 8(2 - 1) \\
&= -283.99 \times 10^2 \text{ Nmm}^3
\end{align*} \]

\[ y_d = \frac{-283.99 \times 10^2}{2 \times 10^5 \times 85 \times 10^6} = -16.7 \text{ mm} \]

Maximum Deflection:

\[ 30x^2 + c_1 - 24(x-1)^2 = 0 \]

\[ 6x^2 + 48x - 187.33 = 0 \]

\[ x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} \]

\[ x = 2.87 \text{ m} \]

\[ F_2 y_{\text{max}} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2 - 1) \]

\[ = 284.67 \text{ kN m}^3 \]

\[ = 284.67 \times 10^{12} \text{ Nmm}^3 \]

\[ y_{\text{max}} = \frac{-287.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \]

\[ = -16.745 \text{ mm} \]
Fig. shows a AB carrying some type of loading and subjected to bending moment as shown in Fig.

Let \( R \) = Radius of curvature of deflected part PQ,

\( d\omega \) = Angle subtended by the area \( P, Q, \) at the centre O.

\( M \) = Bending moment at \( P \) & \( Q \).
\[ h \omega = e \cdot d \omega \]

\[ f(\omega) = d \omega \]

\[ d \omega = R \cdot d \theta \]

\[ d \theta = \frac{d \omega}{R} \]

\[ \frac{M}{I} = \frac{E}{R} \quad \text{(or)} \quad R = \frac{EI}{M} \]

\[ d \omega = \frac{dx}{(\frac{EI}{M})} = \frac{M \cdot dx}{EI} \]

\[ \Theta = \int \frac{M \cdot dx}{EI} = \frac{1}{EI} \int M \cdot dx \]

\[ \Theta_B = \text{Area of B.M dia. } \frac{E}{T} \]

\[ \Theta_B - \Theta_A = \text{Area of B.M. } \frac{EI}{EI} \]

\[ dy = x \cdot d \theta \]

\[ dy = x \cdot \frac{M \cdot dx}{EI} \]

\[ y = \int x \cdot \frac{M dx}{EI} \]

\[ z = \int_0^L x \cdot \frac{M dx}{EI} \]
\[
\frac{y}{EI} = A \times \frac{A}{E} = A \times \frac{\bar{x}}{E}
\]

where: 
- \( A \): Area of B.M dia. b/w A & B.
- \( \bar{x} \): Distance C.G of area A from B.

**Mohr's Theorems:**

5) The change of slope b/w any two pts is equal to the net area of the B.M dia. b/w those points divided by EI.

6) The total deflection b/w any two pts is equal to the moment of the area of B.M dia. b/w the two pts about the last point.

The Mohr's theorems is conveniently used for following cases:

1. Problems on Cantilevers.
2. Simply supported beams carrying symmetrical loading.
3. Beams fixed at both ends.
slope & deflection of SSB carrying a PI load at C:

Using Morin's theorem:

\[ A = \text{Area of B.M diagram between A & \( \frac{c}{2} \)} \]

\[ \frac{1}{2} \times \frac{1}{2} \times \frac{WL}{4} = \frac{WL^2}{16} \]

Slope at A (or) \( \theta_A = \frac{WL^2}{EI} \)

\[ \theta_A = \frac{A_i}{EI} \]

\[ y = \frac{\theta_A x}{EI} \]

\[ x = \frac{WL^2}{16} \]

\[ y = \frac{WL^2}{16} \times \frac{1}{3} = \frac{WL^3}{48EI} \]
\[
\frac{12.87500}{2.7 \times 200 \times 10^6 \times 800 \times 10^{-4}} \times 10^{-3} = 7.94 \text{mm}
\]

Deflection at D, \( y_0 \):

\[
R_b \times 10 - \frac{1}{2} \times 10 \times \frac{1250}{E I} \times 10 / 3
\]

\[= 7.33 \text{mm}\]

Deflection at B, \( y_0 = 0 \) \( R_b \).

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Maxwell's Theorem:

The Maxwell's reciprocal theorem states that:

The work done by the first system of loads due to displacements caused by a second system of loads equals the work done by the second system of loads due to displacements caused by the first system of loads.

**Proof:**

Let \( \text{Point forces} \mathbf{P}_i, i = 1, 2, \ldots, n \) act on an elastic body constrained in a space. Then the strain energy due to this force system is given by:

\[
U_A = \sum_{i=1}^{n} \frac{1}{2} \mathbf{P}_i \mathbf{S}_i
\]

Where \( \mathbf{S}_i \) are the corresponding deflections.
Let point forces \( P_j \), \( j = 1, 2, \ldots m \) be the new set of point forces. Then:

\[ U_B = \sum_{j=1}^{m} \left( P_j \right)_B \delta_j. \]

\[ U_A = \frac{1}{2} \sum_{i=1}^{n} \left( P_i \right)_A \left( \delta_i \right)_A. \]

\[ U_{A,B} = \sum_{i=1}^{n} \left( P_i \right)_A \left( \delta_i \right)_B. \]

\[ U_B = \frac{1}{2} \sum_{j=1}^{m} \left( P_j \right)_B \left( \delta_j \right)_B. \]

\[ U = U_A + U_{A,B} + U_B. \]

\[ U' = U_B + U_{B,A} + U_A. \]

\[ U = U'. \]

\[ U_A + U_{A,B} + U_B = U_B + U_{B,A} + U_A. \]

\[ U_{A,B} = U_{B,A}. \]

\[ \sum_{i=1}^{n} \left( P_i \right)_A \left( \delta_i \right)_B = \sum_{j=1}^{m} \left( P_j \right)_B \left( \delta_j \right)_A. \]