MA 6566 - DISCRETE MATHEMATICS
UNIT - I - LOGICS AND PROOFS

1 - Propositional Logic
2 - Propositional Equivalence
3 - Predicate and Quantifiers
4 - Nested Quantifiers
5 - Rules of Inference
6 - Introduction to Proofs
7 - Proof Methods and Strategy

4 - PROPOSITIONAL LOGIC

SENTENCE
A number of words making complete grammatical structure having sense and meaning is called a sentence. There are two types of sentence:

<table>
<thead>
<tr>
<th>DECLARATIVE SENTENCE</th>
<th>NON-DECLARATIVE SENTENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>▪ Declarative sentence makes a statement that declares something (it) gives reliable information/idea.</td>
<td></td>
</tr>
<tr>
<td>Example:</td>
<td></td>
</tr>
<tr>
<td>(w) Chennai is capital of TN</td>
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<tr>
<td>(w) 1 + 1 = 2</td>
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<tr>
<td>(u) New Delhi is in England</td>
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<tr>
<td>▪ Statement that does not declare something (or) give ideas are called non-declarative sentence.</td>
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<tr>
<td>Example:</td>
<td></td>
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<tr>
<td>▪ Imperative Sentence: Command</td>
<td>Request</td>
</tr>
<tr>
<td>▪ Exclamatory Sentence: !</td>
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<tr>
<td>▪ Interrogative Sentence: ?</td>
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</tbody>
</table>

Proposition (or) Statement:
A proposition is a declarative statement which is either TRUE or FALSE but not both.

Non-Proposition:
Sentences having Command, Questions, Exclamations and neither TRUE/FALSE statement.

Truth Value:
The Truth or Falsity of a proposition is called truth value.

Notations:
▪ If a proposition is true then its truth value is T
▪ If a proposition is false then its truth value is F

P, Q, R, S, ... | P, Q, R, S, ...

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**EXAMPLES OF PROPOSITIONS WITH TRUTH VALUE**

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Truth Value</th>
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</thead>
<tbody>
<tr>
<td>1. Five is a Prime no.</td>
<td>T</td>
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<tr>
<td>2. Dog is a animal</td>
<td>T</td>
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<tr>
<td>3. 3 ÷ 4 = 2</td>
<td>T</td>
</tr>
<tr>
<td>4. The earth is round</td>
<td>T</td>
</tr>
<tr>
<td>5. 1 + 2 = 5</td>
<td>F</td>
</tr>
<tr>
<td>6. Chennai is in England</td>
<td>F</td>
</tr>
<tr>
<td>7. Delhi is Capital of India</td>
<td>T</td>
</tr>
</tbody>
</table>

**EXAMPLES OF NON-PROPOSITIONS**

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. How old are you (?)</td>
<td>Question</td>
</tr>
<tr>
<td>2. What is height of Himalaya (?)</td>
<td>Question</td>
</tr>
<tr>
<td>3. The peacock is very beautiful()</td>
<td>Exclamation</td>
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<tr>
<td>4. What a wonderful joke is this()</td>
<td>Exclamation</td>
</tr>
<tr>
<td>5. Obey my Order. (Command)</td>
<td>Command</td>
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<tr>
<td>6. Please open the door. (Request)</td>
<td>Request</td>
</tr>
<tr>
<td>7. x + 2 = 7 [Neither 'T' or 'F']</td>
<td>Neither 'T' or 'F'</td>
</tr>
</tbody>
</table>

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**PROPOSITIONAL LOGICS**

The area of logic that deals with propositions is called Propositional Logics.

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**PRIMARY STATEMENTS / ATOMIC STATEMENTS**

A declarative sentence which cannot be further split into simple sentences.

Eg: P: Lalitha is a teacher.

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**CONNECTIVES / LOGICAL CONNECTIVES / LOGICAL OPERATORS**

Connectives is an operation which is used to connect two or more than two statements.

Eg: And, or, not.

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**COMPOUND / MOLECULAR / COMPOSITE STATEMENTS**

Statement which contains one or more primary statements and some connectives.

Eg: P: Lalitha is a teacher.
Q: Lalitha teaches Discrete Mathematics.

Compound Statement: Lalitha is a teacher and teaches Discrete Mathematics.

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**TRUTH TABLE**

A truth table displays the relationship between the truth values of propositions.

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**STATEMENT FORMULA**

A statement formula is an expression which is a string consisting of variables, parentheses & connective symbols.
### Five Basic Logical Connectives Based on Precedence of Operators

#### Truth Table

<table>
<thead>
<tr>
<th></th>
<th>NOT (Negation)</th>
<th>AND (Conjunction)</th>
<th>OR (Disjunction)</th>
<th>IF ... then (Conditional)</th>
<th>IF and only If (Biconditional)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Truth Table</strong></td>
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<td><strong>P</strong></td>
<td><strong>Q</strong></td>
<td><strong>P ¬¬¬¬</strong></td>
<td><strong>Q</strong></td>
<td><strong>P → Q</strong></td>
<td><strong>P ↔ Q</strong></td>
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#### Examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
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</table>
| 1       | P: 2 + 5 ≥ 1 (T)  
          | Q: 2 + 5 = 6 (F)  
          | P ∨ Q: 3 ≤ 5 and 2 + 3 = 6 (F)  
| 2       | P: It is raining (T)  
          | Q: I am getting cold (T)  
          | P ∧ Q: It is raining and I am getting cold (T)  
| 3       | P: 3 + 5 = 8 (T)  
          | Q: 5 < 3 (F)  
          | P ∨ Q: 3 + 5 = 8 or 5 < 3 (T)  
| 4       | P: I do not get the money (F)  
          | Q: I shall buy the car (T)  
          | P → Q: If I do not get the money then I shall buy the car (T)  
| 5       | P: You can take the flight (T)  
          | Q: You buy a ticket (F)  
          | P ↔ Q: If you buy a ticket then you can take the flight (F)  
| 6       | P: 2 > 5 (F)  
          | Q: 3 ≤ 4 (T)  
          | P ↔ Q: If 2 > 5 then 3 ≤ 4 (F)  
| 7       | P: 3 + 6 = 10 (T)  
          | Q: 2 + 6 = 9 (F)  
          | P → Q: If 3 + 6 = 10 then 2 + 6 = 9 (F)  
| 8       | P: You can take the flight (T)  
          | Q: I shall buy the car (F)  
          | P ↔ Q: If you can take the flight then I shall buy the car (T)  

#### Further Examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
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</table>
| 9       | P: A is a rational number (T)  
          | Q: √3 is a rational number (F)  
          | P → Q: If A is a rational number then √3 is a rational number (F)  
| 10      | P: 3 + 6 = 9 (T)  
          | Q: 2 + 6 = 9 (F)  
          | P → Q: If 3 + 6 = 9 then 2 + 6 = 9 (T)  

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REMARK: The proposition $P \rightarrow Q$ & $P \leftrightarrow Q$ are usually expressed as follows:

<table>
<thead>
<tr>
<th>$P \rightarrow Q$</th>
<th>$P \leftrightarrow Q$</th>
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</thead>
<tbody>
<tr>
<td>$P$ if $Q$</td>
<td>$P$ if and only if $Q$</td>
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<tr>
<td>$P$ implies $Q$</td>
<td>$P$ iff $Q$</td>
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<tr>
<td>$P$ only if $Q$</td>
<td>$P$ is necessary and</td>
</tr>
<tr>
<td>$Q$ is necessary for $P$</td>
<td>sufficient for $Q$</td>
</tr>
<tr>
<td>$P$ is sufficient for $Q$</td>
<td></td>
</tr>
<tr>
<td>$Q$ whenever $P$</td>
<td>$P$ and $Q$ conversely</td>
</tr>
<tr>
<td>$Q$ provided that $P$</td>
<td></td>
</tr>
<tr>
<td>$Q$ if $P$</td>
<td>$Q$ is implied by $P$</td>
</tr>
</tbody>
</table>

**SYMBOLIZE THE STATEMENTS USING LOGICAL CONNECTIVES**

1. If either Ram takes C++ or Kumar takes Pascal then Latha will take Lotus.
   
   **Solution:** Propositions: $P$: Ram takes C++  
   $Q$: Kumar takes Pascal  
   $R$: Latha will take Lotus.

   **Logical Connectives:** $(P \lor Q) \rightarrow R$

2. Annu can access the internet from campus only if she is a Computer Science major or she is not a Freshgirl.
   
   **Solution:** Propositions: $P$: Annu can access the internet from campus.  
   $Q$: She is a Computer Science major.  
   $R$: She is a Freshgirl.  
   $S$: She is not a Freshgirl.

   **Logical Connectives:** $P \rightarrow (Q \lor \neg S)$

3. (i) If the moon is out & it is not snowing, then Ram goes out for a walk.
   (ii) If the moon is out, then if it is not snowing, Ram goes out for a walk.
   (iii) It is not the case that ram goes out for a walk iff it is not snowing or the moon is out.
   
   **Solution:** Propositions: $P$: The moon is out  
   $Q$: It is snowing  
   $R$: Ram goes out for a walk.

   **Symbolic Expression:**  
   (i) $(P \land \neg Q) \rightarrow R$  
   (ii) $P \rightarrow (\neg Q \rightarrow R)$  
   (iii) $(\neg R) \leftrightarrow (\neg Q \lor P)$
(4) If \( p \): Meenu is rich , \( q \): Meenu is happy , write in symbolic form.
(a) Meenu is poor but happy : \( p \land q \)
(b) Meenu is rich or unhappy : \( p \lor \neg q \)
(c) Meenu is neither rich nor happy : \( p \land \neg q \)
(d) It is necessary for Meenu to be poor in order to be happy : \( q \rightarrow p \)
(e) Meenu to be poor is to be unhappy : \( \neg p \rightarrow \neg q \)
(f) Meenu is rich or he is both poor and unhappy : \( p \lor (\neg p \land \neg q) \)

5) Symbolise the following statements:
(a) If it is raining, then there are clouds in the sky.
(b) If it is not raining, then the sun is not shining and there are clouds in the sky.
(c) The sun is shining if and only if it is not raining.

So: Propositions: \( p \): It is raining
\( q \): There are clouds in the sky
\( r \): The sun is shining

Symbolic expression:
(a) \( p \rightarrow q \)
(b) \( \neg p \lor \neg q \)
(c) \( q \rightarrow \neg p \)

H.W
(a) If it is shining, I shall play tennis in the afternoon.
(b) Finishing the writing of my computer programme before lunch is necessary for playing tennis in this afternoon.
(c) A low boundary and sunshine are sufficient to play Tennis in this afternoon.

Express logical connectives as an English sentence:

5) Let \( p \) and \( q \) be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore" respectively. Express each of these compound propositions as an English sentence.
Sol: Propositions: \( p \): Swimming at the New Jersey shore is allowed
\( q \): Sharks have been spotted near the shore.
(a) \( \neg p \), Sharks have not been spotted near the shore.
(b) \( p \land q \), Swimming at the New Jersey shore is allowed and Sharks have been spotted near the shore.
(c) \( \neg p \lor \neg q \), Swimming at the New Jersey shore is not allowed or Sharks have been spotted near the shore.
(iv) \( P \rightarrow Tq \) → If swimming at the New Jersey shore is allowed then sharks have not been spotted near the shore.
(v) \( Tp \rightarrow P \) → If sharks have not been spotted near the shore then swimming at the New Jersey shore is allowed.
(vi) \( Tp \rightarrow Tq \) → If swimming at the New Jersey shore is not allowed then sharks have not been spotted near the shore.
(vii) \( P \leftrightarrow Tq \) → Swimming at the New Jersey shore is allowed iff sharks have not been spotted near the shore.
(viii) \( Tp \land (P \lor Tq) \) → Swimming at the New Jersey shore is not allowed and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore.

Using the following propositions
\[ P: \text{I finish writing my computer program before lunch.} \]
\[ Q: \text{I shall play tennis in the afternoon.} \]
\[ R: \text{The sun is shining.} \]
\[ S: \text{The boundary is low.} \]
Express the logical connectives as an English sentence.
(i) \( P \rightarrow Q \) → (ii) \( Q \leftrightarrow P \) → (iii) (say) \( Tp \rightarrow Tq \).

<table>
<thead>
<tr>
<th>CONVERSE/CONTRAPOSITIVE AND INVERSE STATEMENTS FOR P &amp; Q Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( P \rightarrow Q ) is a Conditional Statement, then</td>
</tr>
<tr>
<td>(i) Converse of ( P \rightarrow Q )</td>
</tr>
<tr>
<td>(ii) Contrapositive of ( P \rightarrow Q )</td>
</tr>
<tr>
<td>(iii) Inverse of ( P \rightarrow Q )</td>
</tr>
</tbody>
</table>

(i) What are the Contrapositive, the Converse and the Inverse of the implication "The home team wins whenever it is raining".
(ii) Statement: The home team wins whenever it is raining
Modified Statement: If it is raining then home team wins

(i) Contrapositive of \( P \rightarrow Q \): \( \neg Q \rightarrow \neg P \)
   If the home team does not win then it is not raining
(ii) Converse of \( P \rightarrow Q \): \( Q \rightarrow P \)
   If the home team wins then it is raining
(iii) Inverse of \( P \rightarrow Q \): \( \neg P \rightarrow \neg Q \)
   If it is not raining then the home team does not win
2. A positive integer is a prime only if it has no divisors other than 1 and itself.

Sol: Proposition: P: A positive integer is a prime
   q: It has no divisors other than 1 and itself.

(i) Contrapositive of P \( \Rightarrow q \): If it has no divisors other than 1 and itself then a positive integer is not a prime.

(ii) Converse of P \( \Rightarrow q \): If it has no divisors other than 1 and itself then a positive integer is a prime.

(iii) Inverse of P \( \Rightarrow q \): If a positive integer is not a prime then it has divisors other than 1 and itself.

CONSTRUCTION OF TRUTH TABLE

1. How many rows are needed in the truth table of given statement.

Sol: No. of rows needed in the truth table: \( 2^n \) rows, \( n = \log_2 \text{variable} \)

(a) \( p \Rightarrow q \): 1 variable \( \Rightarrow 2^1 = 2 \) rows
(b) \( (p \lor q) \land (q \lor r) \): 4 variables \( \Rightarrow 2^4 = 16 \) rows.
(c) \( q \Rightarrow p \lor q \lor r \): 6 variables \( \Rightarrow 2^6 = 64 \) rows.

2. Construct the truth table for \( (p \land q) \lor (p \land r) \)

Sol: Since the given statement formula consisting of 3 variables, the truth table has \( 2^3 = 8 \) rows.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>( p \land q )</th>
<th>( p \land r )</th>
<th>( (p \land q) \lor (p \land r) )</th>
</tr>
</thead>
<tbody>
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2. Construct the truth table for the compound propositions
   \((P \lor q) \lor \neg (P \land q) \lor \neg (P \land q)\)

   **Sol:** No. of variables : 2 ; No. of rows in truth table : \(2^2 = 4\) rows

<table>
<thead>
<tr>
<th>(P)</th>
<th>(q)</th>
<th>(\neg P)</th>
<th>(\neg q)</th>
<th>((P \lor q))</th>
<th>((P \lor q) \lor \neg (P \land q))</th>
<th>((P \lor q) \lor \neg (P \land q))</th>
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3. Construct the truth table for \((P \rightarrow q) \leftrightarrow (R \leftrightarrow S)\).

   **Sol:** No. of variable : 4 ; No. of rows in Truth table : \(2^4 = 16\) rows

<table>
<thead>
<tr>
<th>(P)</th>
<th>(q)</th>
<th>(R)</th>
<th>(S)</th>
<th>((P \rightarrow q))</th>
<th>((R \leftrightarrow S))</th>
<th>((P \rightarrow q) \leftrightarrow (R \leftrightarrow S))</th>
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### Propositional Equivalence (\(\iff\))

#### Logical Equivalence / Equivalence Rules / Laws of Proposition

<table>
<thead>
<tr>
<th>Law</th>
<th>Rule</th>
</tr>
</thead>
</table>
| 1. **Idempotent Laws** | \(P \land P \iff P\)  
\(P \lor P \iff P\) |
| 2. **Associative Laws** | \((P \land (Q \land R)) \iff (P \land Q) \land R\)  
\((P \lor (Q \lor R)) \iff (P \lor Q) \lor R\) |
| 3. **Commutative Laws** | \(P \land Q \iff Q \land P\)  
\(P \lor Q \iff Q \lor P\) |
| 4. **De Morgan's Laws** | \(\neg (P \land Q) \iff \neg P \lor \neg Q\)  
\(\neg (P \lor Q) \iff \neg P \land \neg Q\) |
| 5. **Distributive Laws** | \(P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)\)  
\(P \lor (Q \land R) \iff (P \lor Q) \land (P \lor R)\) |
| 6. **Complement Laws** | \(P \land \neg P \iff F\)  
\(P \lor \neg P \iff T\) |
| 7. **Dominance Laws** | \(P \land T \iff P\)  
\(P \lor F \iff P\) |
| 8. **Identity Laws** | \(P \land F \iff F\)  
\(P \lor T \iff T\) |
| 9. **Absorption Laws** | \(P \lor (P \land Q) \iff P\)  
\(P \land (P \lor Q) \iff P\) |
| 10. **Double Negation Laws** | \(\neg \neg P \iff P\) |
| 11. **Condition as Disjunction** | \(P \iff Q \iff T \lor Q\) |
| 12. **Biconditional as Conjunction** | \(P \iff Q \iff (P \iff Q) \land (Q \iff P)\) |

#### Tautology, Contradiction, and Contingency

<table>
<thead>
<tr>
<th>Tautology</th>
<th>Contradiction</th>
<th>Contingency</th>
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</thead>
<tbody>
<tr>
<td>Statement Formulae or Resultant Column is always 'True'</td>
<td>Statement Formulae or Resultant Column is always 'False'</td>
<td>Statement Formulae or Resultant Column is always 'True &amp; False'</td>
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**Note:** The table represents the truth values for tautology, contradiction, and contingency in propositional logic.
### Method 1: Construction Using Truth Table

1. Prove that \((p \land (p \rightarrow q)) \rightarrow q\) is a Tautology.

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 proves that \((p \land (p \rightarrow q)) \rightarrow q\) is a Tautology.

2. Prove that \((\neg p \land p) \land q\) is a Contradiction.

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proves that \((\neg p \land p) \land q\) is a Contradiction.

3. Prove that \((p \lor q) \rightarrow (p \lor q)\) is a Contingency.

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proves that \((p \lor q) \rightarrow (p \lor q)\) is a Contingency.

**HW**

1. Show that \((\neg (p \land q)) \land q\) is a Tautology.
2. Show that \((\neg (p \lor q) \land q \rightarrow r) \rightarrow (p \rightarrow q)\) is a Tautology.
3. Verify \((p \lor q) \land q \rightarrow p\) is a Tautology.
4. Determine whether \([(p \land (p \rightarrow q)) \rightarrow q]\) is a Tautology.
5. Determine whether \([(p \lor q) \land (p \rightarrow q)] \rightarrow q\) is a Tautology.
6. Check whether \([(p \rightarrow (q \rightarrow r)) \rightarrow (p \rightarrow (q \rightarrow r))]\) is Tautology.
METHOD 2: TO VERIFY TAUTOLOGY USING EQUIVALENCE RULE

1. Show without constructing truth table for the following.
   a. \((P \lor Q) \land (Q \lor R)\) \(\lor\) \((Q \land R) \land (P \land Q)\) is a tautology.

   Solution:
   \[ (P \lor Q) \land (Q \lor R) \lor (Q \land R) \land (P \land Q) \]
   \[= (P \lor Q) \land (Q \lor R) \lor (Q \land R) \land (P \land Q) \]
   \[= (P \lor Q) \land (Q \lor R) \lor (Q \land R) \land (P \land Q) \]
   \[= (P \lor Q) \land (Q \lor R) \lor (Q \land R) \land (P \land Q) \]
   \[= (P \lor Q) \land (Q \lor R) \lor (Q \land R) \land (P \land Q) \]
   \[= T \]

   *The given statement formula is a tautology.*

2. \((P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q)\) is a tautology.

   Solution:
   \[ (P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q) \]
   \[= (P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q) \]
   \[= (P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q) \]
   \[= (P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q) \]
   \[= (P \land Q) \land (P \lor Q) \lor (P \land Q) \land (P \lor Q) \]
   \[= T \]

   *The given statement formula is a tautology.*

HW:
1. \((P \lor Q) \rightarrow (P \land Q)\) (i) \(P \lor (Q \rightarrow R) \land (P \lor Q)\) (c) \((\neg A \lor (P \rightarrow Q)) \rightarrow P\)

2. DUALITY PRINCIPLE

   In duality, if the compound proposition contains \(T\) \& \(F\) replace by \(F\) \& \(T\) respectively.

   Eq: (i) \(S : (P \land Q) \lor (R \land T)\) (ii) \(S : (F \rightarrow G) \lor (P \land Q) \land (R \rightarrow T)\)

   *: \((P \lor Q) \land (R \land T)\) (ii) \((F \lor G) \land (P \lor Q) \land (R \land T)\)

   *: \((P \lor Q) \land (R \land T)\) (ii) \((F \lor G) \land (P \lor Q) \land (R \land T)\)

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**Logical Equivalence:**

A proposition $p$ is a logical equivalence iff:

1. $p \iff q$ have the same truth value.
2. $p \Rightarrow q$ is a tautology.
3. Assume $p \land q$ (or) Assume $q \land p$ derive $p$.
4. If $LHS = A \land RHS = B$ then $A = B \iff$ Logical Equivalence.

**Problems on Logical Equivalence**

1. Show that $p \iff q \iff \neg p \lor q$.

**Solution:**

**Method:** Constructing a truth table.

Here the truth value of $p \iff q$ and $\neg p \lor q$ are the same.

2. Show that $p \equiv q \iff (p \land q) \lor (q \land p) \equiv (p \lor q) \land (q \lor p)$.

**Solution:**

The truth values of $1, 2, 3$ are the same.

3. Show that $p \Rightarrow q \equiv \neg p \Rightarrow \neg q$.

**Solution:**

METHOD 2: $p \Rightarrow q \iff \neg p \Rightarrow \neg q$ is a tautology.

Assume $p \Rightarrow q \iff \neg p \Rightarrow \neg q$.

$\equiv [\neg p \Rightarrow \neg q] \land [\neg q \Rightarrow \neg p] \equiv (p \Rightarrow q)$. [Simplification]

$\equiv [\neg p \Rightarrow \neg q] \land [\neg q \Rightarrow \neg p] \equiv (p \lor q) \land (q \lor p)$. [Simplification]

$\equiv [\neg p \Rightarrow \neg q] \land [\neg q \Rightarrow \neg p] \equiv (p \lor q) \land (q \lor p)$. [Distributive Law]
\[ (C \lor P) \land (P V T) \land (T V P) \land (T A T) \]

\[ \equiv (C \lor P) \land (T V P) \]

\[ \equiv T \land T \]

\[ \equiv T \]

\[ \overline{R \rightarrow Q} \iff \overline{T V \rightarrow T P} \]

**4.** Show that \( P \rightarrow (Q \rightarrow P) \iff T P \rightarrow (P \rightarrow T Q) \)

**5.** Show that \( P \rightarrow (Q \lor Q) \iff T Y \rightarrow (P \rightarrow Q) \iff [P V T Q] \rightarrow Y \)

**Sol:** Method **\( \Box \): Assume L.H.S \( \land \) derive R.H.S

\[ \text{L.H.S} = P \rightarrow (Q \lor Q) \equiv T P V (Q \lor Q) \]

\[ \equiv (Q \lor Q) \land T P \]

\[ \equiv (Q \lor Q) \land T P \]

\[ \equiv Q \lor (Q \land T P) \]

\[ \equiv T Y \rightarrow (P \rightarrow Q) = \Box \]

\[ \equiv T (P \rightarrow Q) \rightarrow T Y \]

\[ \equiv T (T P V Q) \rightarrow Y \]

\[ \equiv (P \land t q) \rightarrow Y \]

\[ \equiv R.H.S = \Box \]

**6.** Show that \( (T P A (T Q A Y)) \lor (C Q A Y, V C P A Y) ) \iff Y \)

**Sol:** L.H.S = \( (T P A (T Q A Y)) \lor (C Q A Y, V C P A Y) ) \)

\[ \equiv [T P A (T Q A Y)] \lor [C Q A Y] \land V [C P A Y] \land [P Y] \]

\[ \equiv [T P A (T Q A Y)] \lor [C Q A Y] \land V [C P A Y] \land [P q] \]

\[ \equiv [T (P \lor q)] \lor (C P A Y) \land V [C P A Y] \land [P Y] \]

\[ \equiv T A Y \]

\[ \equiv Y \]

\[ \equiv R.H.S \]

**H.W:** Show that

**4.** \( (P \land P) \rightarrow (T P V (T P Q)) \iff (T P V 0) \)

**5.** \( (T (P \rightarrow 0) \iff (C P V 0) \land T (P V 0) \)

**6.** \( (T (C P V 0) A R) \rightarrow (C P V 0 A R) \land (C A R) \)

**9.** \( (C P V 0 V R) \rightarrow (P V V T 0) \land (C P A T V R) \rightarrow (P V R) \)

**11.** \( (P \rightarrow q) \land (P \rightarrow Y) = P \rightarrow (q \land Y) \)

**13.** \( (P \rightarrow Y) \land (Y \rightarrow Y) \)
Show that \( P \implies (q \implies p) \iff T \implies (p \implies q) \)

**Solution:**

**Method 1:** If L.H.S = A & R.H.S = B then \( A \iff B \) (Logical Equivalence)

\[
\begin{align*}
\text{L.H.S} & = P \implies (q \implies p) \\
& = T \lor (T \lor p) \\
& = T \lor T \\
& = T \\
\text{R.H.S} & = T \implies (p \implies q) \\
& = T \land (T \land q) \\
& = T \\
\end{align*}
\]

Here L.H.S = R.H.S, \( \therefore \) The given statement is Equivalent.

Prove the following equivalences by proving the equivalence of the dual \( T \left[ (C \land P \lor A) \lor (C \lor P \land A) \right] \iff T \lor (P \land A) \iff T \)

**Solution:**

**Statement Formula:** \( T \left[ (C \land P \lor A) \lor (C \lor P \land A) \right] \iff T \lor (P \land A) \iff T \)

**Dual Statement Formula:** \( T \left[ C \lor (P \land A) \land (C \lor P \land A) \right] \iff T \)

\[
\begin{align*}
\text{L.H.S} & = T \left[ (C \lor P \land A) \land (C \lor P \land A) \right] \\
& \iff [ (C \lor P \land A) \land (P \land A) ] \land (P \land A) \\
& \iff [ (P \land A) \land (C \lor P \land A) ] \land (P \land A) \\
& \iff [ (P \land A) \land (C \lor P \land A) ] \land (P \land A) \\
& \iff P \land (P \land A) \\
& \iff P \\
& \iff \text{R.H.S} \\
\end{align*}
\]

\( \therefore T \left[ (CP \land A) \land (C \lor P \land A) \right] \iff T \lor (P \land A) \iff T \)

**Work:**

Solve the following using Truth Table or Laws of Logic

\( (a) \ (T \lor P \land q) \iff (q \lor A) \iff (P \lor A) \)

\( (b) \ T \implies (q \implies p) \iff q \implies (p \lor q) \)

\( (c) \ T (p \lor q) \implies p \lor q \iff T \)

\( (d) \ (P \lor q) \land (C \lor P \lor q) \iff q \)

\( (e) \ (C \lor P \lor q) \iff q \iff T \)

\( (f) \ [C \land (P \lor q) \lor (C \land P \lor q)] \iff q \iff T \)

\( (g) \ (T \implies (q \implies p)) \iff q \iff T \)

\( (h) \ (T \implies (q \implies p)) \iff q \iff T \)

\( (i) \ (T \implies (q \implies p)) \iff (p \land A) \iff (p \land A) \iff T \)
Prove that \((p \rightarrow (q ightarrow r)) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)\) by constructing the Truth Table.

\[
\begin{array}{cccccccc}
P & q & r & q \rightarrow r & p \rightarrow (q \rightarrow r) & p \rightarrow q & p \rightarrow r & (p \rightarrow q) \rightarrow (p \rightarrow r) \\
T & T & T & T & T & T & T & T \\
T & T & F & F & F & T & F & F \\
T & F & T & T & T & T & T & T \\
T & F & F & F & F & T & F & F \\
F & T & T & T & T & T & T & T \\
F & T & F & F & T & T & T & T \\
F & F & T & T & T & T & T & T \\
F & F & F & T & T & T & T & T \\
\end{array}
\]

Thus, the given statement formulae is a Tautology Implication.

Show that the following implication's without constructing the T-T:
\[
\begin{align*}
& [(p \lor \neg p) \rightarrow q] \Rightarrow [(p \lor \neg p) \rightarrow r] \Rightarrow (q \rightarrow r) \\
& \text{(Complementary Law)} \\
& \text{(Conditional Law)} \\
& \text{(Dominance Law)} \\
& \text{(Dominance Law)} \\
& \text{(Conditional Law)} \\
& \text{Given} \\
& \text{P \lor \neg P \equiv T} \\
\end{align*}
\]

Show that \((q \rightarrow (p \rightarrow r)) \Rightarrow (r \rightarrow (p \lor \neg p)) \Rightarrow (r \rightarrow q)\).

Thus, the given implication is a Tautology.
\[ \begin{align*}
[\neg q \vee (p \land \neg q)] & \rightarrow [q \rightarrow (p \land \neg q)] \Rightarrow (y \rightarrow q) \\
\equiv [\neg q \rightarrow (p \land \neg q)] \rightarrow (x \rightarrow q) \\
\equiv [\neg q \rightarrow (x \rightarrow q)] \rightarrow (y \rightarrow q) \\
\equiv (\neg q \rightarrow \neg x) \rightarrow (\neg q \rightarrow \neg y) \\
\equiv \neg q \rightarrow \neg x \vee \neg y \\
\equiv \neg y \rightarrow \neg x \vee \neg y \\
\equiv \neg x \vee \neg y \\
\equiv T.
\end{align*} \]

Since \[ [\neg q \rightarrow (p \land \neg q)] \rightarrow [(x \rightarrow q) \rightarrow [q \rightarrow (p \land \neg q)]] \Rightarrow (y \rightarrow q) \] is a Tautology.

\[ \begin{align*}
[(p \rightarrow q) \land (r \rightarrow q)] & \rightarrow [(p \lor q) \rightarrow q].
\end{align*} \]

Show that \[ (p \rightarrow q) \land (r \rightarrow q) \Rightarrow (p \lor q) \rightarrow q. \]

Solution: To prove tautological implication, it is enough to prove \[ [(p \rightarrow q) \land (r \rightarrow q)] \Rightarrow [(p \lor q) \rightarrow q] \] is a tautology.

\[ \begin{align*}
[(p \rightarrow q) \land (r \rightarrow q)] & \Rightarrow [(p \lor q) \rightarrow q] \\
\equiv [(p \rightarrow q) \land (r \rightarrow q)] \rightarrow [(p \lor q) \rightarrow q] \\
\equiv [(p \rightarrow q) \lor (r \rightarrow q)] \rightarrow [(p \lor q) \rightarrow q] \\
\equiv [\neg (p \lor q) \lor (p \lor q)] \rightarrow [(p \lor q) \rightarrow q] \\
\equiv T.
\end{align*} \]

Since \[ [(p \rightarrow q) \land (r \rightarrow q)] \Rightarrow (p \lor q) \rightarrow q \]
\[ \equiv [(p \rightarrow q) \land (r \rightarrow q)] \Rightarrow (p \lor q) \rightarrow q \] is a tautology.
If we write a given statement formula in terms of $\land$, $\lor$ & $\neg$ then it is called **NORMAL FORM / CANONICAL FORM**

<table>
<thead>
<tr>
<th>PRINCIPAL DISJUNCTIVE NORMAL FORM (PDNF)</th>
<th>PRINCIPAL CONJUNCTIVE NORMAL FORM (PCNF)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CONJUNCTION</strong> : $\land$ (PRODUCT)</td>
<td><strong>DISJUNCTION</strong> : $\lor$ (SUM)</td>
</tr>
<tr>
<td>Elementary Product: $\land$</td>
<td>Elementary Sum: $\lor$</td>
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<tr>
<td>The product of variables &amp; their Negation.</td>
<td>The sum of variables and their Negation.</td>
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<tr>
<td>Eq: $P \land Q, Q \land R, P \land Q \land R, P \land Q \land R \land \neg T, P \land Q \land R \land \neg T \land \neg S, \ldots$</td>
<td>Eq: $P \lor Q, Q \lor R, P \lor Q \lor R, P \lor Q \lor R \lor \neg T, P \lor Q \lor R \lor \neg T \lor \neg S, \ldots$</td>
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**Disjunctive Normal Form, DNF:**

$$\text{DNF} = (\text{Elementary}) \lor (\text{Elementary}) \lor (\text{Elementary})$$

$$\text{DNF} = (P \lor Q) \lor (P \lor Q) \lor (P \lor Q) \lor (P \lor Q)$$

**Example:**

$(P \lor Q) \lor (P \lor Q) \lor (P \lor Q) \lor (P \lor Q)$ is an Example.

**Minterms:** Formulas consisting of conjunctions (AND) and (OR) but not both a same variable and its negation.

- Minterms of 2 variables: $2^2 = 4$ terms.
  - $P \land Q, \neg P \land Q, Q \land \neg P, \neg Q \land \neg P$
- Minterms of 3 variables: $2^3 = 8$ terms.
  - $P \land Q \land R, \neg P \land Q \land R, Q \land \neg P \land R, \neg Q \land \neg P \land R, P \land Q \land \neg R, \neg P \land Q \land \neg R, Q \land \neg P \land \neg R, \neg Q \land \neg P \land \neg R$

**PDNF:** Sum of Minterms

Eq: $(P \lor Q) \lor (P \lor Q) \lor (P \lor Q) \lor (P \lor Q)$

**Working Rule to Obtain PDNF:**

1. Write the given statement in terms of $\land$ & $\lor$ only.
2. Apply (Each term) $\land T$.  ($\therefore P \land T = P$)
3. Instead of $T$, apply $P \lor T$.  ($\therefore P \lor T \equiv P$)
4. Apply Distributive Law
5. Apply Commutative Law

**Note:** DNF & CNF are not unique.

**Conjunctive Normal Form, CNF:**

$$\text{CNF} = (\text{Elementary}) \land (\text{Elementary}) \land (\text{Elementary})$$

$$\text{CNF} = (P \land Q) \land (P \land Q) \land (P \land Q) \land (P \land Q) \land (P \land Q)$$

**Example:**

$(P \land Q) \land (P \land Q) \land (P \land Q) \land (P \land Q) \land (P \land Q)$

**Maxterms:** Negative of Minterms.

Eq: Maxterms of 2 variables:

- $P \lor Q, \neg P \lor Q, Q \lor \neg P, \neg Q \lor \neg P$
- Maxterms of 3 variables:

  - $P \lor Q \lor R, \neg P \lor Q \lor R, Q \lor \neg P \lor R, \neg Q \lor \neg P \lor R, P \lor Q \lor \neg R, \neg P \lor Q \lor \neg R, Q \lor \neg P \lor \neg R, \neg Q \lor \neg P \lor \neg R$

**PCNF:** Product of Maxterms

Eq: $(P \lor Q) \land (P \lor Q) \land (P \lor Q) \land (P \lor Q) \land (P \lor Q)$

**Working Rule to Obtain PCNF:**

1. Write the given statement in terms of $\land$ & $\lor$ only.
2. Apply (Each term) $\lor F$.  ($\therefore P \lor F \equiv F$)
3. Instead of $F$, apply $P \land F$.  ($\therefore P \land F \equiv P$)
4. Apply Distributive Law
5. Apply Commutative Law

**Note:** PDNF & PCNF are unique.
(a) Find PDNF and PCNF of the following compound proposition using truth table and laws of propositions.

(\(\neg P \lor \neg Q \land \neg P \lor Q\)) \(\Rightarrow\) (\(P \land Q\))

So: Using truth table

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<td>(\text{(\neg)P}\text{(\lor)Q})</td>
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PDNF: \(\text{\(\neg\)P}\lor\text{\(\neg\)Q}\lor\text{\(\neg\)P}\lor Q\)

PCNF: \((P\lor Q)\land(P\lor Q)\land(\text{\(\neg\)P}\lor Q)\)

Using laws of propositions:

Let \(S = (\text{\(\neg\)P}\lor\text{\(\neg\)Q}) \Rightarrow (P \land Q)\).

\(S = \top(\text{\(\neg\)P}\lor\text{\(\neg\)Q}) \land (P\lor Q)\)

\(\equiv (P\lor Q) \land [\text{\(\neg\)P}\lor Q] \land (P\lor Q)\)

PDNF of \(S = (P\lor Q) \land (P\lor Q) \land (\text{\(\neg\)P}\lor Q)\)

PCNF of \(S = \top(P\lor Q) \land (P\lor Q) \land (\text{\(\neg\)P}\lor Q)\)

PDNF = (P\lor Q) \land (P\lor Q) \land (\text{\(\neg\)P}\lor Q)

PCNF = \(\top(P\lor Q) \land (P\lor Q) \land (\text{\(\neg\)P}\lor Q)\)

(b) \((\text{\(\neg\)P} \Rightarrow \text{\(\neg\)Q}) \land (\text{\(\neg\)Q} \Rightarrow \text{\(\neg\)P})\). Obtain PDNF and hence PCNF.

So: Using truth table

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<td>F</td>
<td>(F)</td>
<td>(P\text{(\land)Q})</td>
<td>(P\text{(\land)Q})</td>
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<tr>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>(T)</td>
<td>(\text{(\neg)P}\text{(\land)Q})</td>
<td>(\text{(\neg)P}\text{(\land)Q})</td>
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</tbody>
</table>

PDNF: \(\text{\(\neg\)P}\lor\text{\(\neg\)Q}\lor\text{\(\neg\)P}\lor Q\)

PCNF: \((P\lor Q)\land(P\lor Q)\land(\text{\(\neg\)P}\lor Q)\)

Enheart: DNF: Sum of elementary product (SP)

PDNF: Sum of miniterms

PCNF: Product of maxterms

PCNF: Product of maxterms.
USING LAWS OF PROPOSITION

Let \( S = (\neg \neg P \rightarrow R) \land (Q \rightarrow P) \)

\[
\begin{align*}
\equiv & (17 \neg P \lor R) \land ([Q \rightarrow P] \land (P \rightarrow Q)) \\
\equiv & (17 \neg P \lor R) \land (Q \lor R) \\
\equiv & (17 \neg P \lor R) \land (Q \lor R) \\
\equiv & (17 \neg P \lor R) \land (Q \lor R) \\
\equiv & (17 \neg P \lor R) \land (Q \lor R) \\
\equiv & (17 \neg P \lor R) \land (Q \lor R) \\
\end{align*}
\]

\( \neg \neg P \lor Q \equiv \neg \neg (P \lor Q) \equiv (P \lor Q) \land (P \lor Q) \equiv P \lor Q \)

\( \neg \neg Q \lor (P \lor Q) \equiv (P \lor Q) \lor Q \equiv (P \lor Q) \lor Q \equiv P \lor Q \)

\( \neg \neg P \lor Q \equiv (P \lor Q) \lor Q \equiv (P \lor Q) \lor Q \equiv P \lor Q \)

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\( \neg \neg P \lor Q \equiv (P \lor Q) \lor Q \equiv (P \lor Q) \lor Q \equiv P \lor Q \)

\( \neg \neg Q \lor (P \lor Q) \equiv (P \lor Q) \lor Q \equiv (P \lor Q) \lor Q \equiv P \lor Q \)
\[\begin{align*}
\text{PCNF, S} &= (\pi v q v r) \wedge (\pi v q v r) \wedge (\pi v q v r) \wedge (\pi v q v r) \\
\text{PCNF, T s} &= (\pi v q v r) \wedge (\pi v q v r) \wedge (\pi v q v r) \\
\text{PDNF, S} &= (\pi v q v r) \wedge (\pi v q v r) \wedge (\pi v q v r) \\
\text{PDNF, T s} &= (\pi v q v r) \wedge (\pi v q v r) \wedge (\pi v q v r)
\end{align*}\]

4. Obtain PDNF & hence PCNF of \((\pi v q) \lor (\pi v q) \lor (\pi v q)\)

Sol: Let \(S = (\pi v q) \lor (\pi v q) \lor (\pi v q)\)

\[\begin{align*}
\text{PDNF, S} &= (\pi v q) \lor (\pi v q) \lor (\pi v q) \\
\text{PCNF, S} &= (\pi v q) \wedge (\pi v q) \wedge (\pi v q) \\
\text{PDNF, T s} &= (\pi v q) \wedge (\pi v q) \wedge (\pi v q) \\
\text{PCNF, T s} &= (\pi v q) \wedge (\pi v q) \wedge (\pi v q)
\end{align*}\]

5. Find PCNF and PDNF of \((\pi \rightarrow (\pi v q)) \land (\pi \rightarrow (\pi v q))\)

Sol: \((\pi \rightarrow (\pi v q)) \land (\pi \rightarrow (\pi v q))\)

\[\begin{align*}
\text{Condition daw} & \\
\text{Distributive law} & \text{Complementary daw} \text{ PVF} = \text{PVF} \\
\text{Distributive law} & \text{PVF} = \text{PVF} \\
\text{Complementary law} & \text{PVF} = \text{PVF} \\
\end{align*}\]
1. Obtain PDNF and hence PCNF of $P \rightarrow (C \rightarrow (\neg p \land q))$

**Solution:**

- $P \rightarrow (C \rightarrow (\neg p \land q))$
- $\equiv P \rightarrow (C \land (\neg p \land q))$
- $\equiv P \rightarrow ((C \land \neg p) \land (C \land q))$
- $\equiv P \rightarrow ([C \land (\neg p)] \land [C \land q])$
- $\equiv P \rightarrow [C \land (\neg p) \land q]$  
- $\equiv P \rightarrow [\neg p \lor q]$  
- $\equiv P \rightarrow (\neg p \lor q)$  
- $\equiv \neg p \lor q$

**PDNF:** $\neg p \lor q$

**PCNF:** $\neg p \lor q$

2. Obtain PDNF for $P \rightarrow (\neg p \land (\neg p \rightarrow q))$

**Solution:**

- $P \rightarrow (\neg p \land (\neg p \rightarrow q))$
- $\equiv P \rightarrow (\neg p \land (\neg p \land q))$
- $\equiv (\neg p \land (\neg p \land q))$
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- $\equiv (\neg p \land (\neg p \land q))$
### 1. HW: Find DNF and CNF for the following statement formulae

(a) \( p \land (p \rightarrow q) \)
(b) \( (p \rightarrow (\neg q \land p)) \land (\neg p \rightarrow (\neg p \lor t)) \)
(c) \( (p \lor q) \land (\neg p \lor q) \)
(d) \( (\neg q \lor r) \lor (\neg q \land t) \lor r \)
(e) \( p \lor (\neg q \land \neg r) \)

### 2. Further Logical Equivalence

1. \((p \land q) \land (p \land q) \equiv (p \lor q) \lor (p \lor q)\)
2. \((p \rightarrow q) \lor (p \rightarrow q) \equiv (p \lor q) \lor (p \lor q)\)
3. \((p \lor q) \land (p \lor q) \equiv p \land (q \lor r)\)

### 3. Rules of Inference

**Theorem:** Premises/Hypothesis: When all premises are assumed to be true, conclusion: Then conclusion is also true.

**Proofs:**
- Direct Proof
- Indirect Proof
- Conditional Proof

### Rules of Inference | Implication Rule | Rules for Valid Conclusion

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
</table>
| 1. Modus Ponens | \( p \rightarrow q \)  
\( p \)  
\( \therefore q \) | \( p, p \rightarrow q \) | \( q \) |
| 2. Modus Tollens | \( p \rightarrow q \)  
\( q \)  
\( \therefore p \) | \( q, p \rightarrow q \) | \( \neg p \) |
| 3. Addition | \( p \)  
\( q \)  
\( \therefore p \lor q \) | \( p, q \) | \( p \lor q \) |
| 4. Conjunction | \( p \)  
\( q \)  
\( \therefore p \land q \) | \( p, q \) | \( p \land q \) |
| 5. Simplification | \( p \land q \)  
\( \therefore p \)  
\( \therefore q \) | \( p \land q \) | \( p \)  
\( \neg q \) |
| 6. Disjunction | \( p \lor q \)  
\( p \lor q \)  
\( \therefore p \)  
\( \therefore q \) | \( p \lor q \) | \( p \lor q \) |
| 7. Hypothetical Syllogism | \( p \rightarrow q \)  
\( q \rightarrow r \)  
\( \therefore p \rightarrow r \) | \( p \rightarrow q \) | \( q \rightarrow r \) |
| 8. Resolution | \( p \lor q \)  
\( \neg p \lor q \)  
\( \therefore q \) | \( p \lor q \) | \( q \lor \neg q \) |
| 9. Dilemma | \( p \rightarrow y, q \rightarrow y \)  
\( p, q \)  
\( \therefore y \) | \( p \lor q \) | \( p \lor q \) |
| 10. | \( p \rightarrow q \)  
\( \therefore p \rightarrow q \) | \( p \rightarrow q \) | \( p \rightarrow q \) |
1. DIRECT PROOF:

When a conclusion is derived from a set of premises by using the equivalence rule and implication rule, then the process of derivation is called a direct proof.

1. Show that TP is a valid conclusion from the premises TPvQ, T(QvR) & TR using logical implication.

 Sol: Premises: TPvQ, T(QvR), TR

 Conclusion: TP [Direct Proof].

<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>T(QvR)</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>TQ &amp; TR</td>
<td>1, Rule T: Demorgan's Law</td>
</tr>
<tr>
<td>3</td>
<td>TQ</td>
<td>2, Rule T: Simplification Law</td>
</tr>
<tr>
<td>4</td>
<td>TPvQ</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>TP</td>
<td>§3.4.3, Rule T: Disjunction Syllogism</td>
</tr>
</tbody>
</table>

TP is a valid conclusion.

2. Show that T is a valid conclusion from the premises P⇒Q, Q⇒R, R⇒S, T & PVT.

 Sol: Premises: P⇒Q, Q⇒R, R⇒S, T & PVT

 Conclusion: T.

<table>
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<tr>
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<th>REASON</th>
</tr>
</thead>
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<tr>
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<td>P⇒Q</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>Q⇒R</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>P⇒R</td>
<td>Rule P</td>
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<tr>
<td>4</td>
<td>R⇒S</td>
<td>Rule P</td>
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<tr>
<td>5</td>
<td>P⇒S</td>
<td>Rule P</td>
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<tr>
<td>6</td>
<td>T</td>
<td>§3.4.3, Rule T: Hypothetical</td>
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<tr>
<td>7</td>
<td>TP</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>PVT</td>
<td>§5.6.3, Rule T: Modus Tollens</td>
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<tr>
<td>9</td>
<td>T</td>
<td>§7.8.3, Rule T: Disjunction</td>
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</tbody>
</table>

T is a valid conclusion.

3. Show that 1S is a valid conclusion from the premises P⇒Q, 1P⇒Q, T(QvR) & SVP.

 Sol: Premises: P⇒Q, 1P⇒Q, T(QvR), SVP

 Conclusion: 1S [Direct Proof].

SVP

TP

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<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neg (q \lor r) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( q \lor r )</td>
<td>( \neg \neg (q \lor r) )</td>
</tr>
<tr>
<td>3</td>
<td>( q \rightarrow r )</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>( p \rightarrow q )</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>( p \rightarrow r )</td>
<td>Rule P</td>
</tr>
<tr>
<td>6</td>
<td>( p \rightarrow q )</td>
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</tr>
<tr>
<td>7</td>
<td>( q \rightarrow p )</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>( p \rightarrow q )</td>
<td>Rule P</td>
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<tr>
<td>9</td>
<td>( \neg s )</td>
<td>Rule P</td>
</tr>
<tr>
<td>10</td>
<td>( s \lor p )</td>
<td>Rule P</td>
</tr>
<tr>
<td>11</td>
<td>( s \lor r )</td>
<td>Rule P</td>
</tr>
<tr>
<td>12</td>
<td>( s \lor p )</td>
<td>Rule P</td>
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... S is a valid conclusion.

Show that \( (p \land q) \land (q \lor r) \) is implied by \( \neg s \) by

1. Premises: \( p \rightarrow q \lor r \) \( q \rightarrow s \) \( r \rightarrow \neg s \)
2. Conclusion: \( s \lor \neg s \)

<table>
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<tr>
<th>STEPS</th>
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<tbody>
<tr>
<td>1</td>
<td>( p \rightarrow q )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( q \rightarrow r )</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>( r \rightarrow s )</td>
<td>Rule P</td>
</tr>
<tr>
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<tr>
<td>5</td>
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<tr>
<td>6</td>
<td>( s \rightarrow r )</td>
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</tr>
<tr>
<td>7</td>
<td>( r \rightarrow \neg s )</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>( s \lor \neg s )</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

... S is a valid conclusion.

Show that \( s \lor r \) is bautologically implied by

\( (p \lor q) \land (p \lor r) \land (q \rightarrow s) \Rightarrow s \lor r \)

1. Premises: \( p \lor q \lor r \) \( q \rightarrow s \) \( r \rightarrow s \)
2. Conclusion: \( s \lor r \)

... S is a valid conclusion.

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<tr>
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<td>2</td>
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<td>Rule P</td>
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<tr>
<td>3</td>
<td>$Q \rightarrow S$</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>$T \lor P \rightarrow S$</td>
<td>Rule P</td>
</tr>
<tr>
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<td>$T \lor S \rightarrow P$</td>
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<td>6</td>
<td>$P \rightarrow R$</td>
<td>Rule P</td>
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<tr>
<td>7</td>
<td>$T \lor S \rightarrow R$</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>$S \lor R$</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

---

"$S \lor R$ is a valid conclusion."

---

5) Show that $R \lor S$ follows tautology from the following premises:

$C \lor D$, $C \lor D \rightarrow T$, $T \rightarrow (A \land T B) \lor (A \land T B) \rightarrow R \lor S$.

Solu: Premises: $C \lor D$, $C \lor D \rightarrow T$, $T \rightarrow (A \land T B) \lor (A \land T B) \rightarrow R \lor S$

Conclusion: $R \lor S$

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<tr>
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</thead>
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<tr>
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<tr>
<td>3</td>
<td>$T \rightarrow (A \land T B)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>$(A \land T B)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$(A \land T B) \rightarrow R \lor S$</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

---

"$R \lor S$ is a valid conclusion."

---

6) Show that $(P \rightarrow Q) \wedge (R \rightarrow S)$, $(Q \wedge M) \wedge (Q \rightarrow N)$, $T \rightarrow (M \wedge N)$, $P \rightarrow R \rightarrow T P$.

Solu: Premises: $(P \rightarrow Q) \wedge (R \rightarrow S)$, $(Q \wedge M) \wedge (Q \rightarrow N)$, $T \rightarrow (M \wedge N)$, $P \rightarrow R$.

Conclusion: $T P$.

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<th>REASON</th>
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<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>$P \rightarrow Q$</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>$R \rightarrow S$</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>$(Q \wedge M) \wedge (Q \rightarrow N)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$Q \wedge M$</td>
<td>Rule P</td>
</tr>
<tr>
<td>6</td>
<td>$Q \rightarrow N$</td>
<td>Rule P</td>
</tr>
<tr>
<td>7</td>
<td>$T \rightarrow (M \wedge N)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>$P \rightarrow R$</td>
<td>Rule P</td>
</tr>
<tr>
<td>9</td>
<td>$P \rightarrow S$</td>
<td>Rule P</td>
</tr>
<tr>
<td>10</td>
<td>$P \rightarrow N$</td>
<td>Rule P</td>
</tr>
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</table>
8. Show that the hypothesis \((pq) \lor r\) and \(r \Rightarrow s\) imply \(PVS\):

Premises: \((pq) \lor r\) and \(r \Rightarrow s\)

Conclusion: \(PVS\)

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<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
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</thead>
<tbody>
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<td>1</td>
<td>((pq) \lor r)</td>
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<td>2</td>
<td>((pq) \lor r) \lor (pq)</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>((pq) \lor r) \lor (pq) \lor r</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>(r \Rightarrow s)</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>(PVS)</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

9. "If the labour market is perfect, then the wages of all persons in a particular employment are equal. But it is a case that wages for such persons are not equal. Therefore the labour market is not perfect."

Test validity of the argument.

Solution: Let \(P\): Labour market is perfect

\(~q\): Wages of all persons in a particular employment.

Premises: \(P \Rightarrow q\)

Conclusion: \(\neg P\)

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<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
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</thead>
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<td>1</td>
<td>(P \Rightarrow q)</td>
<td>Rule P</td>
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<tr>
<td>2</td>
<td>(\neg q)</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>(P)</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

\(1, 2, 3\): Rule T: Modus Ponens
If there was a cricket, then travelling was difficult. If they arrived on time then travelling was not difficult. They arrived on time. Therefore there was no cricket. Show that proposition is valid argument.
Sol: Let P: There was a cricket
   Q: Travelling was difficult. R: Travelling was not difficult.
Premises: $p \rightarrow q$, $r \rightarrow q$, $r$
Conclusion: $\neg p$.

<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>CONCLUSION</th>
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<tbody>
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<td>$r \rightarrow q$</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>$r$</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>$\neg q$</td>
<td>$\vdash 123$, Rule T: Modus Tollens</td>
</tr>
<tr>
<td>4</td>
<td>$p \rightarrow q$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$\neg p$</td>
<td>$\vdash 133$, Rule T: Modus Tollens</td>
</tr>
</tbody>
</table>

The given proposition forms a valid conclusion.

Test the validity of the argument:
If the music party could not play music or the refreshment were not delivered on time, then the new year's party would have been cancelled and the organiser Rammu would have been angry. If the party were cancelled, then refunds would have to be made. No refunds we made. Therefore music party could play music.
Sol: Let P: Music party could play music
   Q: The refreshment were delivered on time.
   R: The new year party were cancelled.
   S: The organiser Rammu was angry
   T: The funds has to be made.
Premises: $(\neg p \lor \neg q) \rightarrow (\neg s)$, $r \rightarrow t$, $\neg t$.
Conclusion: $p$.

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<td>$r \rightarrow t$</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>$\neg r$</td>
<td>$\vdash 123$, Rule T: Modus Tollens</td>
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<td>4</td>
<td>$t \lor \neg s$</td>
<td>$\vdash 133$, Rule T: Addition, $p \rightarrow p q$</td>
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<td>$s \lor p \neg s$</td>
<td>$\vdash 163$, Rule T: De Morgan's Law</td>
</tr>
<tr>
<td>6</td>
<td>$(\neg p \lor \neg q) \rightarrow (p q)$</td>
<td>$\vdash 165$, Rule T: De Morgan's Law</td>
</tr>
<tr>
<td>7</td>
<td>$(p \lor \neg q)$</td>
<td>$\vdash 172$, Rule T: De Morgan's Law</td>
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<tr>
<td>8</td>
<td>$p$</td>
<td>$\vdash 172$, Rule T: De Morgan's Law</td>
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</table>
### 1. INDIRECT METHOD

For doing indirect method introduce the negation of the conclusion as a additional premise and from the additional premise together with the given premise derive conclusion.

#### Using indirect method of Proof, Prove that $P \rightarrow R, \neg S, PVQ \rightarrow SYR$

**Solution:**

- **Premises:** $P \rightarrow R, \neg S, PVQ$
- **Conclusion:** $SYR$ (Indirect Proof)
- **Additional Premise:** $\neg (SYR) = TS \land TR = \neg (Conclusion)$

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<td>$\neg 13$, Rule T: Simplification</td>
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<td>3</td>
<td>$TR$</td>
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<tr>
<td>4</td>
<td>$P \rightarrow R$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$TP$</td>
<td>$\neg 133$, Rule T: Modus Tollens</td>
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<tr>
<td>6</td>
<td>$PVQ$</td>
<td>Rule P</td>
</tr>
<tr>
<td>7</td>
<td>$Q$</td>
<td>$\neg 653$, Rule T: Disjunction</td>
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<td>8</td>
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<tr>
<td>9</td>
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<td></td>
<td>$P$</td>
<td>$\neg 103$, Rule T: Complement Law</td>
</tr>
</tbody>
</table>

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### 2. USING INDIRECT METHOD SHOW THAT $P \rightarrow Q, Q \rightarrow R, PVY, \neg (PVY) \rightarrow Y$

**Solution:**

- **Premises:** $P \rightarrow Q, Q \rightarrow R, PVY$
- **Conclusion:** $Y$ (Additional Premises: $\neg (Conclusion) = \neg Y$)

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<td>( T )</td>
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<tr>
<td>2</td>
<td>( P V R )</td>
<td>RuleP.</td>
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<td>3</td>
<td>( P )</td>
<td>( #2,17 ), RuleT: Disjunctive Syllogism.</td>
</tr>
<tr>
<td>4</td>
<td>( P \rightarrow Q )</td>
<td>RuleP.</td>
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<td>5</td>
<td>( Q )</td>
<td>( #4,33 ), Modus Poneos: RuleT.</td>
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<td>( Q \rightarrow R )</td>
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<td>( R )</td>
<td>( #5,53 ), RuleT: Modus Poneos.</td>
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<td>8</td>
<td>( T (P \lor R) )</td>
<td>RuleP.</td>
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<td>9</td>
<td>( TP \lor TV )</td>
<td>RuleT: Demorgan's Law.</td>
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<td>( P \rightarrow TV )</td>
<td>( #9,75 ), RuleT: Conditional Law.</td>
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<td>11</td>
<td>( TP )</td>
<td>( #8,73 ), RuleT: Modus Tollens</td>
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<tr>
<td>12</td>
<td>( PA \lor TP )</td>
<td>( #3,11 ), RuleT: Conjunction.</td>
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<tr>
<td>13</td>
<td>( F )</td>
<td>RuleT: Complement Law.</td>
</tr>
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</table>

\( \therefore \) \( y \) is a valid conclusion.

**Using Indirect Method of Proof:** Derive \( P \rightarrow 7S \) from \( P \rightarrow (OVR) \), \( \alpha \rightarrow TP \), \( S \rightarrow TR \), \( P \).

**Solution:** Premises: \( P \rightarrow (OVR) \), \( \alpha \rightarrow TP \), \( S \rightarrow TR \), \( P \).

**Conclusion:** \( P \rightarrow 7S \).

**Steps**

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<td>( S )</td>
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<td>( OVR )</td>
<td>( #4,12 ), RuleT: Modus Poneos.</td>
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<td>( TR )</td>
<td>( #6,13 ), RuleT: Modus Poneos</td>
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<td>( \alpha )</td>
<td>( #5,73 ), RuleT: Disjunctive Syllog.</td>
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<td>( P )</td>
<td>( #10,83 ), RuleT: Conjunction Law</td>
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</tr>
<tr>
<td>13</td>
<td>( F )</td>
<td>RuleT: Complement Law</td>
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</table>

\( \therefore P \rightarrow 7S \) is a valid conclusion.
Show that the following implication by using indirect method
R → Tq, RVs, S → Tq, P → q → 7p.

S1: Premises: R → Tq, RVs, S → Tq, P → q

Conclusion: 7p [Indirect Proof]

Additional Premises: T (7p) ≡ P

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<tr>
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<td>RVs → Tq</td>
<td>S4153, Rule T: Equivalence Rule</td>
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<td>q ∧ Tq</td>
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<td>q</td>
<td>S93, Rule T: Complement Law</td>
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</table>

• 7p is a valid conclusion.

How
0. Prove by indirect method (7p, P → q, RVs → t).
2. Using indirect method, show that P → q, q → R, RVs → R.
3. Use indirect method and prove that
(a) R → Tq, RVs, S → Tq, P → q → 7p.
(b) P → q, R → q, S → (P ∨ R), S → q.
(c) E → S, S → H, A → q → H → T (EBA).

CONDITIONAL PROOF

If the conclusion is of the form P → q, then set
Additional Premise: P 5 Conclusion: q

0. Show that R → S can be derived from P → (q → s), TRV and s.

S1: Premises: P → (q → s), TRV, s

Additional Premises: R 5 Conclusion: s

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<tr>
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<td>TRV</td>
<td>s</td>
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<td>Rule P</td>
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<td>P → (q → s)</td>
<td>S2113, Rule T: Disjunction</td>
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<td>0</td>
<td>q → s</td>
<td>Rule P</td>
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<td>s</td>
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<td>s</td>
<td>S5163, Rule T: Modus Ponens</td>
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</tbody>
</table>
4. Derive $P \rightarrow (Q \rightarrow S)$ using the Rule CP from $P \rightarrow (Q \rightarrow R)$, $Q \rightarrow (R \rightarrow S)$.

**Solution:**
- **Premises:** $P \rightarrow (Q \rightarrow R)$, $Q \rightarrow (R \rightarrow S)$
- **Conclusion:** $P \rightarrow (Q \rightarrow S)$  [Conditional Proof]

**Additional Premises:** $P$

**Final Conclusion:** $Q \rightarrow S$

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<th>Reason</th>
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<tr>
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<td>$Q \rightarrow (R \rightarrow S)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>$Q \rightarrow (Q \rightarrow S)$</td>
<td>Rule 2, 3, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>5</td>
<td>$(Q \rightarrow Q) \rightarrow S$</td>
<td>Rule S17</td>
</tr>
<tr>
<td>6</td>
<td>$Q \rightarrow (Q \rightarrow S)$</td>
<td>Rule P</td>
</tr>
<tr>
<td>7</td>
<td>$P \rightarrow (Q \rightarrow S)$ is a valid conclusion.</td>
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</tr>
</tbody>
</table>

5. Show that the hypothesis "If you send me an email message, then I will finish writing the program." If you don't send me an email message then I will go to sleep early, and if I go to sleep early then I will wake up feeling refreshed."

**Solution:**
- $P$: You send me an email message
- $Q$: I will finish writing the program
- $R$: I will go to sleep early
- $S$: I will wake up feeling refreshed

**Premises:** $P \rightarrow Q$, $1 \rightarrow P \rightarrow R$, $R \rightarrow S$

**Conclusion:** $Q \rightarrow S$  [Conditional Proof]

**Additional Premises:** $Q$

**Final Conclusion:** $S$

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<th>Conclusion</th>
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<tbody>
<tr>
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<td>Rule AP</td>
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<tr>
<td>2</td>
<td>$P \rightarrow Q$</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>$R \rightarrow S$</td>
<td>Rule 2, 3, Rule T: Modus Tollens</td>
</tr>
<tr>
<td>4</td>
<td>$Q \rightarrow R$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$R \rightarrow S$</td>
<td>Rule 2, 3, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>6</td>
<td>$S$</td>
<td>Rule 2, 3, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>7</td>
<td>$Q \rightarrow S$</td>
<td>Final Conclusion</td>
</tr>
</tbody>
</table>

6. `STUDENTSFOCUS.COM Conclusion`
1. **Show that** \( R \rightarrow S \) **can be derived from the Premises** \( P \rightarrow G \rightarrow S \) **and** \( F \). **Prove using** Conditional **Proof**. (a) \( P \rightarrow G \rightarrow S \) \( \rightarrow P \rightarrow (P \land S) \)  
(b) \( P \land S \rightarrow R \) \( \rightarrow (P \land S) \rightarrow R \)  
(c) \( P, P \rightarrow (Q \rightarrow R) \rightarrow Q \rightarrow S \)  

**Consistency and Inconsistency of Premises**

A set of formulae \( H_1, H_2, \ldots, H_m \) is said to be

- **INCONSISTENT** if \( H_1 \land H_2 \land H_3 \land \ldots \land H_m \land \neg R \land \neg T \land \neg E \)
- **CONSISTENT** if \( H_1 \land H_2 \land H_3 \land \ldots \land H_m \land R \land T \land E \)

**1.** Show that \( P \rightarrow Q \land P \rightarrow R \land Q \rightarrow T \) and \( P \) are inconsistent.

**SoL:** Premises: \( P \rightarrow Q \land P \rightarrow R \land Q \rightarrow T \land P \)  

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( P \rightarrow Q )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( P \rightarrow R )</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>( Q \rightarrow T )</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>( P \rightarrow Q \land P \rightarrow R \Rightarrow \neg \neg R \land \neg \neg T )</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>( R \rightarrow T )</td>
<td>Rule P</td>
</tr>
<tr>
<td>6</td>
<td>( P \rightarrow R )</td>
<td>Rule P</td>
</tr>
<tr>
<td>7</td>
<td>( Q \rightarrow T )</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>( R \rightarrow T )</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

"Given set of premises are inconsistent."

**2.** Prove that the Premises \( a \rightarrow (b \rightarrow c) \), \( d \rightarrow (b \land c) \), \( \neg c \), \( \neg d \) are inconsistent.

**SoL:** Premises: \( a \rightarrow (b \rightarrow c) \), \( d \rightarrow (b \land c) \), \( \neg c \), \( \neg d \)  

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a \land d )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( a \land d )</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>( b \land d )</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>( a \rightarrow (b \rightarrow c) )</td>
<td>Rule P</td>
</tr>
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<td>( b \rightarrow c )</td>
<td>Rule P</td>
</tr>
<tr>
<td>6</td>
<td>( d \rightarrow (b \land c) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>7</td>
<td>( \neg c )</td>
<td>Rule P</td>
</tr>
<tr>
<td>8</td>
<td>( \neg d )</td>
<td>Rule P</td>
</tr>
</tbody>
</table>

"Given set of premises are inconsistent."

**3.** Show that the following premises are inconsistent.

- If Jack misses many classes through illness, he fails high school.
- If Jack fails high school, then he is uneducated.
- If Jack reads a lot of books, then he is not uneducated.
(iv) Jack misses many classes through illness & reads a lot of books.

301: Let P: Jack misses many classes through illness
    Q: Jack fails high school
    R: Jack is uneducated
    S: Jack reads a lot of books

Premises: P → Q, Q → R, S → T, P ↔ S

<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P → Q</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>Q → R</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>P → T</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>S → T</td>
<td>Rule P, Hypothetical Syll.</td>
</tr>
<tr>
<td>5</td>
<td>T → S</td>
<td>Rule P</td>
</tr>
<tr>
<td>6</td>
<td>P → T</td>
<td>Rule T, Contrapositive</td>
</tr>
<tr>
<td>7</td>
<td>T P V T</td>
<td>Rule T, Hypothetical Syll.</td>
</tr>
<tr>
<td>8</td>
<td>T (P ∧ S)</td>
<td>Rule T, Conditional Law</td>
</tr>
<tr>
<td>9</td>
<td>(P ∧ S) ∧ T (P ∧ S)</td>
<td>Rule T, Complement Law</td>
</tr>
<tr>
<td>10</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

Given set of premises are inconsistent.

641: Show that the following premises are inconsistent.

(a) If Rama gets his degree, he will go for a job. If he goes for a job, he will get married soon. If he goes for higher studies, he will not get married. Rama gets his degree and goes for higher studies.

(b) A diagnostic message is stored in a buffer or it is retransmitted. A diagnostic message is not stored in the buffer. If a diagnostic message is stored in the buffer, then it is retransmitted. A diagnostic message is not transmitted.
## Predicate Calculus

**Predicate**: A part of a declarative sentence that attributes a property to the subject.

Example: Statement function/propositional function, \( P(x) : x \) is a boy

- **Statement**: \( P(\text{Ram}) : \text{Ram is a boy} \)

### Proposition

<table>
<thead>
<tr>
<th>Place</th>
<th>Predicate</th>
<th>Place</th>
<th>Predicate</th>
<th>Place</th>
<th>Predicate</th>
<th>Place</th>
<th>Predicate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Variable: 1</td>
<td>Predicate: 1</td>
<td>Variable: 2</td>
<td>Predicate: 1</td>
<td>Variable: 3</td>
<td>Predicate: 1</td>
<td>Variable: 4</td>
<td>Predicate: 1</td>
</tr>
<tr>
<td>P(x): Ram is a boy</td>
<td>P(y): John is a boy</td>
<td>P(x, y): Ram and John are taller than y</td>
<td>P(x, y, z): Ram and John played bridge</td>
<td>P(x, y, z): Ram and John played bridge</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Propositional Functions: Problems

1. Let \( P(x) \) denote the statement "\( x > 3 \)". What are the truth values of \( P(4) \) & \( P(2) \)?
   - \( P(4) = \text{true} \), \( P(2) = \text{false} \).

2. Let \( a(x, y) \) denote "\( x = y + 3 \)". What are the truth values of the proposition \( a(1, 2); a(3, 10) \)?
   - \( a(1, 2) = \text{false} \), \( a(3, 10) = \text{false} \).

3. Let \( R(x, y, z) \) denote "\( x + y = z \)". What are the truth values for the proposition \( R(1, 2, 3); R(-2, 1, 3) \)?
   - \( R(1, 2, 3) = \text{true} \), \( R(-2, 1, 3) = \text{false} \).

### Simple Statement Functions

- Defined to be an expression consisting of a predicate symbol and an individual variable.
- When the variable is replaced by the name of any object.

Example:

- Statement: \( P(x) : x \) is a teacher.
- Statement: \( T(\text{John}) : \text{John is a teacher} \)

### Compound Statement Functions

- Obtained by combining one or more simple statement functions using logical connectives.

Example:

- \( M(x) : x \) is a man.
- \( H(x) : x \) is a mortal.

### Quantifiers

- Quantifiers is one which is used to quantify the nature of variables such as "all, some, none or one".

- To create proposition from a proposition function.

- The universe of discourse specifies the possible value of the variable.
UNIVERSAL QUANTIFIERS

The expression "All" is the Universal quantifier. We denote it by (∀x).

The following phrases have the same meaning as "All".
1. For all x
2. For every x
3. For each x
4. Everything x s.t.
5. Each thing x is such that

Examples:
1. (∀x) x is a dog
2. (∀x) x has a tail
3. (∀x)[D(x) ∧ T(x)]

EXISTENTIAL QUANTIFIERS

The expression "Some" is the Existential quantifier. We denote it by (∃x).

The following phrases have the same meaning as "Some".
1. For some x
2. Some x s.t.
3. There exists x s.t.
4. There is x s.t.
5. There is at least one x such that

Examples:
1. (∃x) x is a dog
2. (∃x)[D(x) ∧ T(x)]

PROBLEMS UNDER UNIVERSE OF DISCOURSE

(2) Let P(x) be the statement "x > 7". What is the truth value of the quantification (∀x) P(x), where the universe of discourse consists of all real numbers.

Solution:
Given: P(x) : x > 7
Universe of discourse: All real numbers

To find Truth Value: (∀x) P(x)

P(x) is true for all real no.: x < 7

(3) Let Q(x) : x < 2. What is the truth value of the quantification (∃x) Q(x), where the universe of discourse consists of all real numbers.

Solution:
Given: Q(x) : x < 2
Universe of discourse: All real numbers

To find Truth Value: (∃x) Q(x)

Q(x) is not true for every real no.: x ≥ 2

(4) Let P(x) : x > 3. What is the truth value of (∃x) P(x), where the universe of discourse consists of all real numbers?

Solution:
Given: P(x) : x > 3
Universe of discourse: All real numbers

To find Truth Value: (∃x) P(x)

P(x) is true when x = 4

(5) Let A = {1, 2, 3, 4, 5, 6}. Determine the truth value of (∃x)(x ∈ A) ∧ (x > 2.5)

Solution:
P(x) : x > 2.5
Universe of discourse: A = {1, 2, 3, 4, 5, 6}

To find Truth Value: (∃x)(x ∈ A) ∧ (x > 2.5)

(∃x)(x ∈ A) ∧ (x > 2.5) is true since x = 3, 4 > 2.5

(6) Find the truth value of (x) (P(x) → Q(x)) ∨ (x) (R(x) ∧ x > 4) where the universe of discourse E being E = {2, 3, 4, 5}.

Solution:
P: x > 1
Q(x) : x > 3
R(x) : x > 4

(∃x) (P(x) → Q(x)) ∨ (x) (R(x) ∧ x > 4)
To find the truth value: 
\((x \rightarrow (p \rightarrow \neg (\neg p \land \neg q)) \land \neg (\exists x) R(x))\).

**P is true and \(q(1)\) is false.**

\[\therefore (x) (p \rightarrow q(x)) \text{ is false.}\]

\[\therefore (\forall x) R(x) \text{ is false [\(\forall (2)\), \(\forall (3)\), \(\forall (4)\) are all false].}\]

\[\therefore (x) (p \rightarrow q(x)) \land (\exists x) R(x) \text{ is true [\(\forall \), \(VF=E\)].}\]

6. Let the universe of discourse be \(E = \{5, 6, 7, 8\}\). Let \(A = \{5, 6\}\) and \(B = \{6, 7, 8\}\). Let \(P(x) : x \text{ is in } A \land q(x) : x \text{ is in } B \land R(x, y) : \{x+y < 12\}\). Find the truth value of

\[\therefore \exists x (P(x) \rightarrow q(x)) \rightarrow R(5, 6) \land x.\]

So: \(\exists x (P(x) : x \text{ is in } A \land q(x) : x \text{ is in } B ; B = \{6, 7, 8\})\).

\(R(x, y) : x+y < 12\)

To find the truth value:
\[\forall x (\exists x) (P(x) \rightarrow q(x)) \rightarrow R(5, 6) [\text{stb} < 12].\]

\[\therefore \exists x (P(x) \rightarrow q(x)) \rightarrow R(5, 6) \text{ is true.}\]

HW

6. Let \(P(x) : x \text{ is an even integer } \land R(x, y) : x \text{ is divisible by } y.\)

Let the universe of discourse be the set \(U = \{1, 2, 3, 4, 5, 6\}\). Find the truth values of the following:

(a) \(P(16) \land P(4) \land R(4, 2) \land R(16, 2)\)

2. Let \(P(x) : x \leq 32\) and \(q(x) : x \text{ is a multiple of } 10\). Let the variable be the set of all positive integers. Find the value of \(\exists x : P(x) \rightarrow q(x)\).

3. Let \(A = \{1, 2, 3, 4, 5\}\). Determine the truth value of each of the following:

(i) \(\exists x \in A \land x^2 = 4\) (iv) \(\forall x \in A \land x^3 \leq 130\) (v) \(\forall x \in A \land x^2 - 3x - 10\)

---

**FREE AND BOUND VARIABLES**

<table>
<thead>
<tr>
<th><strong>BOUND VARIABLE</strong></th>
<th><strong>FREE VARIABLE</strong></th>
<th><strong>SCOPE OF VARIABLE</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>The variable is said to be bound if it is concerned with either universal ((\forall x)) or existential ((\exists x)) quantifier.</td>
<td>The variable which is not concerned with any quantifier is called free variable.</td>
<td>The scope of the quantifier is the formula immediately following the quantifier.</td>
</tr>
</tbody>
</table>

**EXAMPLES:**

Find the scope of the quantifiers and the nature of occurrence of the variable of the formula.

(i) \((\forall x) P(x, y) : \text{ Bound variable: } x \land \text{ Free variable: } y \land \text{ Scope of quantifier: } P(x, y)\).
2. \( \forall x \ P(x) \land \exists x' \) : Bound Variable: \( x \) in \( P(x) \)
   Free Variable: \( x \) in \( \forall x' \)
   Scope of quantifier: \( P(x) \)

3. \( \exists x \ (P(x) \lor Q(x)) \) : Bound Variable: \( x \)
   Free Variable: 
   Scope of quantifier: \( P(x) \lor Q(x) \)

4. \( (\exists x \ P(x) \rightarrow (\exists y \ R(x,y))) \) : Bound Variable: \( x \) in \( \exists x \)
   \( y \) in \( \forall y \)
   Scope of quantifier: \( P(x) \rightarrow (\exists y \ R(x,y)) \)
   Scope of \( \exists x \) : \( C(x) \lor \forall x \)

**SM\( \overline{X} \) NEGATING QUANTIFIED EXPRESSION**

1. \( \neg \exists x \ P(x) = \forall x \neg P(x) \)
2. \( \neg \forall x \ P(x) = \exists x \neg P(x) \)

**EXAMPLES:** Negate the following statements.

(a) \( P \) : Every student in your class has taken a course in Calculus.

   \( \neg P \) : It is not the case that every student in your class has taken a course in Calculus.

   \( \neg P \) : Some students in your class has not taken a course in Calculus.

(b) \( P \) : No one has done every problem in the exercise.

   \( \neg P \) : Someone has done every problem in the exercise.

(c) \( Q \) : Some People who trust others are rewarded.

   \( \neg Q \) : Every People who trust others are not rewarded.

**EXAMPLES:** Negating a Proposition

(d) \( P \) : To enter into the country you need a passport or a voter registration card.

   \( \neg P \) : To enter into the country you need not have a passport or a voter registration card.

**TRANSLATE THE FOLLOWING INTO LOGICAL EXPRESSIONS-SYMBOLIC**

\[ \text{If then = Every ; and = Some} \]

1. Every student in this class has studied Calculus.

   \( \forall x \ \text{Student} \rightarrow (\exists x \ P(x)) \)

2. Restatement: For every person \( x \), if \( x \) is a student in this class then \( x \) has studied Calculus.

   Proposition : \( S(x) \) : \( x \) is a student in this class

   Function : \( C(x) \) : \( x \) has studied Calculus

   Symbolic Form : \( \forall x \ [S(x) \rightarrow C(x)] \)
Some students in the class have visited Mexico.
Every student in this class has visited either Canada or Mexico.

Sol: Restatement:

1. For some \( x \), \( x \) is a student in this class and has visited Mexico.
2. For every \( x \), if \( x \) is a student in this class then \( x \) has visited either Canada or Mexico.

Propositional function:
- \( S(x) \): \( x \) is a student in the class
- \( C(x) \): \( x \) has visited Canada
- \( M(x) \): \( x \) has visited Mexico

Symbolic form:
1. \( \exists x \ (S(x) \land M(x)) \)
2. \( \forall x \ (S(x) \rightarrow (C(x) \lor M(x))) \)

3. All lions are fierce

Sol: Restatement:

1. For every \( x \), if \( x \) is a lion then \( x \) is fierce.
2. For some \( x \), \( x \) is a lion and does not drink coffee.
3. For some \( x \), \( x \) is a fierce creature and does not drink coffee.

Propositional function:
- \( L(x) \): \( x \) is a lion
- \( F(x) \): \( x \) is fierce

Symbolic form:
1. \( \forall x \ (L(x) \rightarrow F(x)) \)
2. \( \exists x \ (L(x) \land 
\neg F(x)) \)
3. \( \exists x \ (L(x) \land F(x)) \)

4. Let \( g(x,y) \): \( x \) is taller than \( y \).

Translate the following into formula:
For any \( x \) and \( y \), if \( x \) is taller than \( y \), then it is not true that \( y \) is taller than \( x \).

Sol: Propositional function:
- \( g(x,y) \): \( x \) is taller than \( y \).

Symbolic form:
(\( \forall x \) \( \forall y \) \( \neg (g(y,x) \land y \neq x) \))

Nested quantifier: Two quantifiers are nested if one is within the scope of the other. E.g.: \( \forall x \exists y \ (x+3y=0) \)

5. All the world loves a lover.

Sol: Restatement:

1. For all \( x, y \), if \( x \) is a person and \( y \) is a lover then \( x \) loves \( y \).

Propositional function:
- \( P(x) \): \( x \) is a person
- \( L(y) \): \( y \) is a lover
- \( R(x,y) \): \( x \) loves \( y \).

Symbolic form:
\( \forall x \ (P(x) \land L(y) \rightarrow R(x,y)) \)

STUDENTSFOCUS.COM
It is not true that all roads lead to Denmark.

Restatement: It is false that for all x, y, x is a road then x leads to Denmark.

Proposition: \( R(x) \land x \text{ is } R \land D(x) \rightarrow x \text{ leads to Denmark.} \\
Symbolic form: \( \exists x \in \mathbb{R} \forall y \geq x \land R(x) \land D(x) \rightarrow x \text{ leads to Denmark.} \)

No one has done every problem in the exercise.

Restatement: There is no one who has done every problem in the exercise.

Proposition: \( \exists x \forall y \exists z \land x \text{ has done problem } y \).

Symbolic form: \( \exists x \forall y \exists z \land x \text{ has done problem } y \).

Negating:

\( \neg \exists x \forall y \exists z \land x \text{ has done problem } y \).

\( \forall y \exists z \exists x \land x \text{ has not done problem } y \).

Write the symbolic form of:

x is the father of the mother of y.

Proposition: \( P(x) \land x \text{ is the father of } y \land y \text{ is the mother of } z \).

Symbolic form: \( \exists z \land P(x) \land z < x \land z < y \).

Negating:

\( \neg \exists z \land P(x) \land z < x \land z < y \).

\( \forall z \colon \neg (P(x) \land z < x \land z < y) \).

Given any positive integer, there is a greater positive integer.

Proposition: \( x \in \mathbb{Z}^+ \land P(x) \land \exists y \land y > x \).

Symbolic form: \( x \in \mathbb{Z}^+ \land x < y \).

Negating:

\( \neg \exists y \land x < y \).

\( \forall y \land x < y \).

Given any positive integer, there is a greater positive integer.

Proposition: \( \exists x \land P(x) \land \forall y \land y < x \).

Symbolic form: \( \exists x \land P(x) \land x > y \).

Negating:

\( \neg \exists x \land P(x) \land x > y \).

\( \forall x \land \neg (P(x) \land x > y) \).

Given any positive integer, there is a greater positive integer.

Proposition: \( \exists x \land P(x) \land \forall y \land y < x \).

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Negating:

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\( \forall x \land \neg (P(x) \land x > y) \).

Given any positive integer, there is a greater positive integer.

Proposition: \( \exists x \land P(x) \land \forall y \land y < x \).

Symbolic form: \( \exists x \land P(x) \land x > y \).

Negating:

\( \neg \exists x \land P(x) \land x > y \).

\( \forall x \land \neg (P(x) \land x > y) \).
@ Write the symbolic form and negate the following statement.

1. Everyone who's healthy can do all kinds of work.
   Sol: \( \forall x : (x \text{ healthy person}) \land \forall y : y \text{ can do } y. \)
   Symbolic: \( A : (\forall x y) \left( (C x (x y) \land H (y)) \rightarrow D (x y) \right) \)

   Negation: \( \neg A : (\exists x (C x y) \land H (y)) \lor D (x y) \)

2. Some people are not admired by everyone.
   Sol: \( \exists x : x \text{ is a person} \land \exists y : y \text{ is admired by } y \).
   Symbolic: \( A : \exists x (C x y) \land (P y \rightarrow F (x y)) \)

   Negation: \( \neg A : \forall x (C x y) \lor \forall y (P y \land F (x y)) \)

3. Everyone should help his neighbour or his neighbour will not help him.
   Sol: Proposition function: \( P (x y) : x \text{ is a person} \land F (x y) : x \text{ is admired by } y \).
   Symbolic form: \( G (x y) : x \text{ helps } y \).

   Negation: \( \neg A : \forall x (C x y) \rightarrow \exists y (C (x y) \land \exists y (P y)) \)

4. Everyone agrees with someone and someone agrees with everyone.
   Sol: Proposition function: \( B (x y) : x \text{ agree with } y \).
   Symbolic form: \( A : \exists x \exists y (B (x y) \land A (y x)) \)

   Negation: \( \neg A : \exists x (B (x y)) \lor \exists y (B (y x)) \)

---

Translate the statement to symbolic form.

1. The sum of two positive integers is always positive.
   Sol: \( \forall x \forall y (C x > 0 \land C y > 0) \rightarrow (C (x + y) > 0) \)

2. Every real no except zero has a multiplication inverse.
   Sol: \( \forall x (C x > 0 \rightarrow \exists y (C y = 1)) \)

3. The product of a positive real no and a negative real no is always a negative real no.
   Sol: \( \forall x (C x > 0 \land C y < 0) \rightarrow (C (x y) < 0) \)

4. If a person is female and is a parent then the person is someone's mother.
   Sol: \( \forall x (C f x \land C p x) \rightarrow \exists y (C m (x y)) \).

5. If any one is good then John is good.
   Sol: If there is one \( x \) such that \( x \) is a person \( S x \) is good then John is good.
   Sol: \( \exists x (C p x \land G (x)) \rightarrow G (\text{John}) \text{ where } G (x) : x \text{ is good} \).

6. He is ambitious or no one is ambitious.
   Sol: \( \exists x (C p x \land G (x)) \lor \neg (\text{any } x \text{ is a person}) \rightarrow (\forall x (C p x) \rightarrow \neg G (x)) \).
6) Every one who likes fun will enjoy each of these plays.
Sol: \( \forall x (x \text{ likes fun} \land p(x)) \rightarrow \forall y: x \text{ will enjoy } y \\
\forall x [\forall x (x \text{ likes fun} \\ p(x)) \rightarrow \forall y: x \text{ will enjoy } y] \)
### Steps and Premises

<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\forall x (H(x) \rightarrow M(x)) \equiv (\forall x) H(x) \rightarrow (\forall x) M(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>$H(c) \rightarrow M(c)$</td>
<td>Rule vs.</td>
</tr>
<tr>
<td>3</td>
<td>$\exists x H(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>4</td>
<td>$H(c)$</td>
<td>Rule Eq.</td>
</tr>
<tr>
<td>5</td>
<td>$M(c)$</td>
<td>$\exists x \forall y (x = y)$, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>6</td>
<td>$\exists x M(x)$</td>
<td>Rule Eq.</td>
</tr>
</tbody>
</table>

#### Additional Notes

2. **Prove that** $\forall x (P(x) \rightarrow Q(x))$, $\forall x (R(x) \rightarrow \neg Q(x))$.

   **Solution:**

   - Premises: $\forall x (P(x) \rightarrow Q(x))$, $\forall x (R(x) \rightarrow \neg Q(x))$
   - Conclusion: $\forall x (R(x) \rightarrow \neg Q(x))$. [Direct Proof | Rule (P)]

<table>
<thead>
<tr>
<th>STEPS</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\forall x (P(x) \rightarrow Q(x)) \equiv (\forall x) P(x) \rightarrow (\forall x) Q(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>$P(c) \rightarrow Q(c)$</td>
<td>Rule vs.</td>
</tr>
<tr>
<td>3</td>
<td>$\forall x (R(x) \rightarrow \neg Q(x))$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>4</td>
<td>$R(c) \rightarrow \neg Q(c)$</td>
<td>Rule vs.</td>
</tr>
<tr>
<td>5</td>
<td>$\neg Q(c)$</td>
<td>$\exists x \forall y (x = y)$, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>6</td>
<td>$\neg Q(c)$</td>
<td>Rule Eq.</td>
</tr>
</tbody>
</table>

3. **Show that** $\forall x (P(x) \lor Q(x))$, $\forall x \neg P(x) \rightarrow \forall x Q(x)$.

   **Solution:**

   - Premises: $\forall x (P(x) \lor Q(x))$, $\forall x \neg P(x) \rightarrow \forall x Q(x)$
   - Conclusion: $\forall x Q(x)$. [Direct Proof]

<table>
<thead>
<tr>
<th>STEPS</th>
<th>PREMISES</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\forall x [P(x) \lor Q(x)] \equiv \forall x P(x) \lor \forall x Q(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>$P(c) \lor Q(c)$</td>
<td>Rule vs.</td>
</tr>
<tr>
<td>3</td>
<td>$\forall x \neg P(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>4</td>
<td>$\neg P(c)$</td>
<td>Rule vs.</td>
</tr>
<tr>
<td>5</td>
<td>$Q(c)$</td>
<td>$\exists x \forall y (x = y)$, Rule T: Modus Ponens</td>
</tr>
<tr>
<td>6</td>
<td>$\neg Q(c)$</td>
<td>Rule Eq.</td>
</tr>
</tbody>
</table>

4. **Prove** $\exists x (P(x) \land Q(x)) \rightarrow \exists x P(x) \land \exists x Q(x)$.

   **Solution:**

   - Premises: $\exists x (P(x) \land Q(x))$
   - Conclusion: $\exists x P(x) \land \exists x Q(x)$. [Direct Proof]

<table>
<thead>
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<tr>
<td>1</td>
<td>$\exists x (P(x) \land Q(x)) \equiv \exists x P(x) \land \exists x Q(x)$</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>$P(c) \land Q(c)$</td>
<td>Rule vs.</td>
</tr>
</tbody>
</table>

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6. Show that \( \forall x \, (p(x) \Rightarrow (q(x) \land r(x))) \), \( \exists x \, (p(x) \land s(x)) \Rightarrow \exists x \, (q(x) \land r(x)) \).

Solution:

Premises: \( \forall x \, (p(x) \Rightarrow (q(x) \land r(x))) \), \( \exists x \, (p(x) \land s(x)) \).

Conclusion: \( \exists x \, (q(x) \land s(x)) \).

Direct Proof:

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \forall x , (p(x) \Rightarrow (q(x) \land r(x))) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( p(c) \Rightarrow (q(c) \land r(c)) )</td>
<td>Rule US</td>
</tr>
<tr>
<td>3</td>
<td>( \exists x , (p(x) \land s(x)) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>4</td>
<td>( p(c) \land s(c) )</td>
<td>Rule ES</td>
</tr>
<tr>
<td>5</td>
<td>( q(c) \land r(c) )</td>
<td>Rule T: Simplification</td>
</tr>
<tr>
<td>6</td>
<td>( q(c) )</td>
<td>Rule T: Simplification</td>
</tr>
<tr>
<td>7</td>
<td>( r(c) )</td>
<td>Rule T: Simplification</td>
</tr>
<tr>
<td>8</td>
<td>( r(c) \land s(c) )</td>
<td>Rule T: Modus Ponens</td>
</tr>
<tr>
<td>9</td>
<td>( \exists x , (q(x) \land s(x)) )</td>
<td>Rule T: Conjunction</td>
</tr>
</tbody>
</table>

6. Show that \( \forall x \, (p(x) \land r(x)) \Rightarrow (q(x) \land s(x)) \).

Solution:

Premises: \( \forall x \, (p(x) \land r(x)) \Rightarrow (q(x) \land s(x)) \), \( \forall x \, (p(x) \land r(x)) \).

Conclusion: \( \forall x \, (q(x) \land s(x)) \).

Direct Proof:

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \forall x , (p(x) \land r(x)) \Rightarrow (q(x) \land s(x)) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>( q(x) \land s(x) )</td>
<td>Rule US</td>
</tr>
<tr>
<td>3</td>
<td>( \forall x , (p(x) \land r(x)) \Rightarrow (q(x) \land s(x)) )</td>
<td>Rule US</td>
</tr>
<tr>
<td>4</td>
<td>( q(x) \land s(x) )</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>( r(x) )</td>
<td>Rule T: Modus Tollens</td>
</tr>
</tbody>
</table>

\( \exists \) nested quantifiers leftmost quantifier is to be applied first then the next.

5. Show that \( (\exists x) \, (f(x) \Rightarrow g(x)) \) follows logically from

(a) \( (\exists x) \, (f(x) \land s(x)) \Rightarrow (y) \, (M(y) \Rightarrow w(y)) \)

(b) \( \forall y \, (M(y) \land w(y)) \Rightarrow \forall (y) \rightarrow \)
### Direct Proof

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ y \in M(y) \land \lnot w(y) ]</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>[ M(c) \land \lnot w(c) ]</td>
<td>Rule E1.</td>
</tr>
<tr>
<td>3</td>
<td>[ \lnot (M(c) \lor w(c)) ]</td>
<td>Rule E2.</td>
</tr>
<tr>
<td>4</td>
<td>[ \lnot (M(c) \rightarrow \lnot w(c)) ]</td>
<td>Rule T: Condition Law.</td>
</tr>
</tbody>
</table>
| 5     | \[ \lnot [M(c) 
\rightarrow \lnot w(c)] \] | Rule U1. |
| 6     | \[ \lnot [M(c) 
\rightarrow \lnot w(c)] \] | Rule T: Condition Law. |
| 7     | \[ \exists x \left[ F(x) \land s(x) \right] \rightarrow [y] \left[ M(y) 
\rightarrow \lnot w(y) \right] \] | Rule P. |
| 8     | \[ \exists x \left[ F(x) \land s(x) \right] \] | Rule T: Mediation. |
| 9     | \[ (x) \lnot [F(x) \land s(x)] \] | Rule US. |
| 10    | \[ \lnot [F(x) \land s(x)] \] | Rule T: Simplification. |
| 11    | \[ \lnot F(x) \lor \lnot s(x) \] | Rule: Condition Law. |
| 12    | \[ \lnot F(c) \lor \lnot s(c) \] | Rule E1. |
| 13    | \[ \exists x \left[ F(x) \rightarrow s(x) \right] \] | | |

**Conclusion:** \[ \exists x \left[ F(x) \rightarrow s(x) \right] \] is a valid conclusion.

### Indirect Proof

Additional Premises: \[ \lnot [(x) p(x) \lor \exists x q(x)] \]

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>[ (x) \lnot p(x) \land (x) \lnot q(x) ]</td>
<td>Rule AP.</td>
</tr>
<tr>
<td>2</td>
<td>[ (x) \lnot p(x) ]</td>
<td>Rule T: Simplification.</td>
</tr>
<tr>
<td>3</td>
<td>[ (x) \lnot q(x) ]</td>
<td>Rule T: Simplification.</td>
</tr>
<tr>
<td>4</td>
<td>[ \lnot p(a) ]</td>
<td>Rule E5.</td>
</tr>
<tr>
<td>5</td>
<td>[ \lnot q(a) ]</td>
<td>Rule US.</td>
</tr>
<tr>
<td>6</td>
<td>[ (x) p(x) \lor (x) q(x) ]</td>
<td>Rule P.</td>
</tr>
<tr>
<td>7</td>
<td>[ p(a) \lor q(a) ]</td>
<td>Rule US.</td>
</tr>
<tr>
<td>8</td>
<td>[ \lnot p(a) \land \lnot q(a) ]</td>
<td>Rule T: Demorgan.</td>
</tr>
<tr>
<td>9</td>
<td>[ \lnot (p(a) \lor q(a)) ]</td>
<td>Rule T: Demorgan.</td>
</tr>
<tr>
<td>10</td>
<td>[ (p(a) \lor q(a)) \land \lnot (p(a) \lor q(a)) ]</td>
<td>Rule T: Complem.</td>
</tr>
</tbody>
</table>

**Conclusion:** \[ (x) p(x) \lor \exists x q(x) \]
Use of Rule and obtain the following implication

\[(\forall x \ (P(x) \rightarrow \exists x' (x' = x))) \land (\forall x \ (R(x) \rightarrow \exists x' (x' = x))) \rightarrow (\forall x \ (R(x) \rightarrow \exists x' (x' = x)))\]

Sol: Premises: \((\forall x \ (P(x) \rightarrow \exists x' (x' = x)))\) and \((\forall x \ (R(x) \rightarrow \exists x' (x' = x)))\)

Conclusion: \((\forall x \ (R(x) \rightarrow \exists x' (x' = x)))\)

Additional Premises: \((\forall x \ R(x))\)

Final Conclusion: \((\forall x \ P(x))\).

<table>
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<tr>
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</tr>
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<tbody>
<tr>
<td>1</td>
<td>((\forall x \ R(x)))</td>
<td>(\text{Rule UN}^1)</td>
</tr>
<tr>
<td>2</td>
<td>(R(a))</td>
<td>(\text{Rule US}^1)</td>
</tr>
<tr>
<td>3</td>
<td>((\forall x \ (P(x) \rightarrow \exists x' (x' = x))))</td>
<td>(\text{Rule AP}^1)</td>
</tr>
<tr>
<td>4</td>
<td>(P(a) \rightarrow \exists x' (x' = a))</td>
<td>(\text{Rule US}^1)</td>
</tr>
<tr>
<td>5</td>
<td>((\forall x \ (R(x) \rightarrow \exists x' (x' = x))))</td>
<td>(\text{Rule UK}^1)</td>
</tr>
<tr>
<td>6</td>
<td>(R(a) \rightarrow \exists x' (x' = a))</td>
<td>(\text{Rule US}^1)</td>
</tr>
<tr>
<td>7</td>
<td>(\exists x' (x' = a))</td>
<td>(\text{Rule TL}^1)</td>
</tr>
<tr>
<td>8</td>
<td>(P(a))</td>
<td>(\text{Rule UG}^1)</td>
</tr>
<tr>
<td>9</td>
<td>((\forall x \ P(x)))</td>
<td>(\text{Rule TL}^1)</td>
</tr>
</tbody>
</table>

\((\forall x \ (R(x) \rightarrow \exists x' (x' = x)))\) is a valid conclusion.

Show that the Premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Mala is a student in this class" imply the conclusion "Mala has taken a course in computer science." 

Sol: Proposition: \(P(x)\): \(x\) is in this discrete mathematics class, \(P(c)\): Mala is a student in this class, \(C(x)\): \(x\) has taken a course in computer science, \(C(c)\): Mala has taken a course in computer science.

Premises: \((\forall x \ (P(x) \rightarrow C(x))) \land P(c)\)

Conclusion: \(C(c)\).

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((\forall x \ (P(x) \rightarrow C(x))) \land P(c))</td>
<td>(\text{Rule AP}^1)</td>
</tr>
<tr>
<td>2</td>
<td>(P(c))</td>
<td>(\text{Rule US}^1)</td>
</tr>
<tr>
<td>3</td>
<td>(C(c))</td>
<td>(\text{Rule UP}^1)</td>
</tr>
<tr>
<td>4</td>
<td>(C(c))</td>
<td>(\text{Rule TL}^1)</td>
</tr>
</tbody>
</table>

Show that the Premises "A student in this class has not read the book" and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book".

Sol: Proposition: \(P(x)\): \(x\) is a student in this class, \(q(x)\): \(x\) has read the book.
Show that the following premises are valid. All integers are rational numbers. Some integers are power of 2. Therefore some rational number are power of 2.

Sol: Proposition:
- \( p(x) \): \( x \) is an integer.
- \( q(x) \): \( x \) is a rational number.
- \( r(x) \): \( x \) is a power of 2.

Premises:
- \( \forall x (p(x) \rightarrow q(x)) \)
- \( \exists x (p(x) \land r(x)) \)

Conclusion:
- \( \exists x (q(x) \land r(x)) \)

Show that the following argument is valid. "Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both serial and parallel port.

Sol: Proposition:
- \( p(x) \): \( x \) is a microcomputer.
- \( q(x) \): \( x \) has a serial interface port.
- \( r(x) \): \( x \) has a parallel port.
### Premises & Conclusion

**Premises:**
- \( \forall x [P(x) \implies q(x)] \)
- \( \exists x [P(x) \land r(x)] \)

**Conclusion:**
- \( \exists x [P(x) \land (q(x) \land r(x))] \)

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### Steps & Reasoning

<table>
<thead>
<tr>
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<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \forall x [P(x) \implies q(x)] )</td>
<td>Rule P.</td>
</tr>
<tr>
<td>2</td>
<td>( P(x) \implies q(x) )</td>
<td>( \exists x [P(x) \land r(x)] )</td>
</tr>
<tr>
<td>3</td>
<td>( \exists x [P(x) \land r(x)] )</td>
<td>Rule P.</td>
</tr>
<tr>
<td>4</td>
<td>( P(x) \land r(x) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>5</td>
<td>( r(x) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>6</td>
<td>( q(x) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>7</td>
<td>( q(x) \land r(x) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>8</td>
<td>( (q(x) \land r(x)) \land q(x) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>9</td>
<td>( P(x) \land (q(x) \land r(x)) )</td>
<td>Rule Es.</td>
</tr>
<tr>
<td>10</td>
<td>( \exists x [P(x) \land (q(x) \land r(x))] )</td>
<td>Rule Es.</td>
</tr>
</tbody>
</table>

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### Test the Validity of the Following Argument:

If an integer is divisible by 10, then it is divisible by 2.
If an integer is divisible by 2, then it is divisible by 3.
Therefore, the integer divisible by 10 is also divisible by 3.

**Proposition:**
- \( D_{10}(x) : x \text{ is divisible by } 10 \)
- \( D_2(x) : x \text{ is divisible by } 2 \)
- \( D_3(x) : x \text{ is divisible by } 3 \)

**Premises:**
- \( \forall x [D_{10}(x) \implies D_2(x)] \)
- \( \forall x [D_2(x) \implies D_3(x)] \)

**Conclusion:**
- \( \forall x [D_{10}(x) \implies D_3(x)] \)

---

### Show that the Premises "One student in this class knows how to write programs in JAVA" and "Everyone knows how to write programs in JAVA can get a high paying job." imply the Conclusion "Someone in the class can get a high paying job."

**Propositions:**
- \( A(x) : x \text{ is in the class} \)
- \( B(x) : x \text{ knows JAVA programming} \)
- \( H(x) : x \text{ can get a high paying job} \)

**Premises:**
- \( \forall x [A(x) \land B(x) \implies H(x)] \)
- \( \exists x [A(x) \land B(x)] \)

**Conclusion:**
- \( \exists x [A(x) \land B(x) \implies H(x)] \)
### Verification of the Argument

1. **Premises:**
   - $\exists x \ [A(x) \land B(x)]$
   - $A(c \land B(c))$
   - $A(c)$
   - $B(c)$
   - $[B(x) \rightarrow H(x)]$
   - $B(c) \rightarrow H(c)$
   - $H(c)$
   - $[A(c) \land H(c)]$
   - $\exists x \ [A(x) \land H(x)]$

2. **Reasons:**
   - Rule P.
   - Rule ES.
   - $\exists x \ [A(x) \land B(x)]$
   - Rule E.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule E.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.

3. **Conclusion:**
   - Therefore, $A(c)$.
   - $H(c)$.
   - $B(c)$.
   - $\exists x \ [A(x) \land H(x)]$

---

### Exercise 1

1. **Premises:**
   - $\forall x \ [A(x) \rightarrow B(x)]$
   - $B(c)$
   - $A(c)$
   - $C(c)$

2. **Conclusion:**
   - $[A(c) \rightarrow B(c)]$
   - $[B(c) \rightarrow C(c)]$

3. **Reasons:**
   - Rule P.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.

---

### Exercise 2

1. **Premises:**
   - $\forall x \ [A(x) \rightarrow B(x)]$
   - $B(c)$
   - $A(c)$
   - $C(c)$

2. **Conclusion:**
   - $[A(c) \rightarrow B(c)]$
   - $[B(c) \rightarrow C(c)]$

3. **Reasons:**
   - Rule P.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.

---

### Exercise 3

1. **Premises:**
   - $\exists x \ [A(x) \land B(x)]$
   - $A(c \land B(c))$
   - $A(c)$
   - $B(c)$
   - $[B(x) \rightarrow H(x)]$
   - $B(c) \rightarrow H(c)$
   - $H(c)$
   - $[A(c) \land H(c)]$
   - $\exists x \ [A(x) \land H(x)]$

2. **Reasons:**
   - Rule P.
   - Rule ES.
   - $\exists x \ [A(x) \land B(x)]$
   - Rule E.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule E.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.

---

### Exercise 4

1. **Premises:**
   - $\exists x \ [A(x) \land B(x)]$
   - $A(c \land B(c))$
   - $A(c)$
   - $B(c)$
   - $[B(x) \rightarrow H(x)]$
   - $B(c) \rightarrow H(c)$
   - $H(c)$
   - $[A(c) \land H(c)]$
   - $\exists x \ [A(x) \land H(x)]$

2. **Reasons:**
   - Rule P.
   - Rule ES.
   - $\exists x \ [A(x) \land B(x)]$
   - Rule E.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule E.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.

---

### Exercise 5

1. **Premises:**
   - $\forall x \ [A(x) \rightarrow B(x)]$
   - $B(c)$
   - $A(c)$
   - $C(c)$

2. **Conclusion:**
   - $[A(c) \rightarrow B(c)]$
   - $[B(c) \rightarrow C(c)]$

3. **Reasons:**
   - Rule P.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.

---

### Exercise 6

1. **Premises:**
   - $\forall x \ [A(x) \rightarrow B(x)]$
   - $B(c)$
   - $A(c)$
   - $C(c)$

2. **Conclusion:**
   - $[A(c) \rightarrow B(c)]$
   - $[B(c) \rightarrow C(c)]$

3. **Reasons:**
   - Rule P.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.
   - Rule US.

---

### Exercise 7

1. **Premises:**
   - $\exists x \ [A(x) \land B(x)]$
   - $A(c \land B(c))$
   - $A(c)$
   - $B(c)$
   - $[B(x) \rightarrow H(x)]$
   - $B(c) \rightarrow H(c)$
   - $H(c)$
   - $[A(c) \land H(c)]$
   - $\exists x \ [A(x) \land H(x)]$

2. **Reasons:**
   - Rule P.
   - Rule ES.
   - $\exists x \ [A(x) \land B(x)]$
   - Rule E.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule E.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
   - $\exists x \ [B(x) \rightarrow H(x)]$
   - Rule US.
   - $\exists x \ [A(x) \land H(x)]$
   - Rule US.
**THEOREM:** A theorem is a statement that can be shown to be true.

**Proof:** A proof is a valid argument that establishes the truth of a mathematical statement.

1. **Direct Proof:** A direct proof shows that a conditional statement \( P \implies Q \) is true by showing that if \( P \) is true then \( Q \) must be true.

2. **Proofs by Contradiction:** In proof by contradiction or reductio ad absurdum, we take \( \neg Q \) as a hypothesis and using axioms, definitions, together with rules of inference shows that \( \neg P \) must be false.

3. **Proofs by Contrapositive:** In proof by contrapositive of \( P \implies Q \), we assume the conclusion \( Q \) is false and then by rules of inference, we reach a contradiction. Our assumption \( \neg Q \) is false, is wrong. Thus indirectly we prove \( P \implies Q \).

4. **Vacuous Proof:** The vacuous proof of \( P \implies Q \) is to show that \( P \) is false.

5. **Trivial Proof:** The trivial proof of \( P \implies Q \) is to show that \( Q \) is true.

6. **Proofs of Equivalence:** To prove a theorem of the form \( P \iff Q \), we show that \( P \implies Q \) and \( Q \implies P \) are both true.

7. **Counter Examples:** A statement of the form \( \forall x P(x) \) is false. We need only to find a counter example that is an example \( x \) for which \( P(x) \) is false.

---

**SUMMARY OF PROOFS**

<table>
<thead>
<tr>
<th>Methods of Proofs</th>
<th>Assumptions / Hypothesis (A)</th>
<th>Derive</th>
<th>Conclusion (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Direct Proof</strong></td>
<td>( P \implies Q ) = True</td>
<td>( P ) True</td>
<td>( Q ) True</td>
</tr>
<tr>
<td><strong>Indirect Proof</strong></td>
<td>( P \implies \neg Q ) \iff ( \neg P \implies \neg Q )</td>
<td>( \neg Q ) True</td>
<td>( \neg P ) True</td>
</tr>
<tr>
<td><strong>Other Proof</strong></td>
<td>Vacuous Proof, ( P \implies Q )</td>
<td>( P ) False</td>
<td>( Q ) False</td>
</tr>
<tr>
<td><strong>Proof of Equivalence</strong></td>
<td>( P \iff Q ) \iff ( A \iff B )</td>
<td>( A ) True</td>
<td>( B ) False</td>
</tr>
<tr>
<td><strong>Counter Example</strong></td>
<td>( \forall x P(x) ) \iff ( \neg \exists x Q(x) )</td>
<td>Find ( x ) for which ( P(x) ) is false</td>
<td>( \neg \exists x Q(x) )</td>
</tr>
</tbody>
</table>
TYPE 1: PROBLEMS BASED ON DIRECT PROOFS.

6. Give a direct proof of the statement: “The square of an odd integer is an odd integer.”

SOL: Given: “The square of an odd integer is an odd integer.” (or)
“If n is an odd integer, then \( n^2 \) is an odd integer.”

Theorem: \( \forall n \left( P(n) \Rightarrow Q(n) \right) \) where \( P(n) : n \) is an odd.

Proof: Direct Proof: \( P \Rightarrow Q \) TRUE. \( Q(n) : n^2 \) is an odd.

Hypothesis: Assume that \( P \) is TRUE i.e. \( n \) is an odd int. is true.

Analysis: By definition of an odd integer, \( n = 2k + 1, k \) int.

\[ n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \]

Conclusion: R.H.S = \( 2(2k^2 + 2k) + 1 \) is not divisible by 2.

\[ \Rightarrow n^2 \text{ is not divisible by 2} \]

\[ \Rightarrow n^2 \text{ is an odd integer} \]

\[ \Rightarrow P \Rightarrow Q \text{ is TRUE.} \]

Thus, if \( n \) is an odd integer, then \( n^2 \) is an odd integer.

7. Give a direct proof of “The sum of two odd integers is even.”

SOL: Given: If \( n \) is odd and \( m \) is odd then \( n + m \) is an even integer.

Theorem: \( \forall n \left( P(n) \Rightarrow Q(n) \right) \) where \( P(n) : n \) is odd inte & \( m \) is odd.

Proof: Direct Proof: \( P \Rightarrow Q \) TRUE. \( Q(n) : n + m \) is even

Hypothesis: Assume that \( P \) is true i.e. \( n \) is odd & \( m \) is odd.

Analysis: \( n = 2k + 1 \) for some \( k \) & \( m = 2l + 1 \) for some \( l \)

\[ n + m = (2k + 1)(2l + 1) = 2(k + l + 1) \]

Conclusion: R.H.S = \( 2(k + l + 1) \) is divisible by 2.

\[ \Rightarrow n + m \text{ is even integer.} \]

\[ \Rightarrow P \Rightarrow Q \text{ is TRUE.} \]

Thus, if \( n \) is odd and \( m \) is odd then \( n + m \) is even.

8. Use a direct proof to show that “Every odd integer is the difference of two squares.”

SOL: Given: If \( n \) is an odd integer then \( n = s^2 - t^2 \).

Theorem: \( \forall n \left( P(n) \Rightarrow Q(n) \right) \) where \( P(n) : n \) is odd inte.

Proof: Direct Proof: \( P \Rightarrow Q \) TRUE. \( Q(n) : n = s^2 - t^2 \)

Hypothesis: Assume that \( P \) is true.

Analysis: If \( n \) is an odd integer then \( n = 2k + 1 \) where \( k \) is some integer.

\[ n = k^2 + (2k + 1) \quad \text{for some } k \]

\[ n = (2k + 1)^2 - k^2 \]

\[ n = (k + 1)^2 - k^2 = s^2 - t^2, \quad s = k + 1, \quad t = k \]

Conclusion: We observe that \( n = s^2 - t^2 \).

\[ \Rightarrow P \Rightarrow Q \text{ is TRUE.} \]

Hence, every odd integer is the difference of two squares.
2. Using direct proof, prove that the sum of two rational numbers is rational.

**Solution:** Given: If \( a \) is a rational number and \( b \) is a rational number, then \( a + b \) is a rational number.

**Theorem:** \( \forall (a(n) \rightarrow b(n)) \), \( P(n) \) is a rational number and \( \exists b(n) \) rational number.

**Proof:** DIRECT PROOF: \( P \Rightarrow Q \Rightarrow \boxed{\text{TRUE}} \)

**Hypothesis:** Assume \( P \Rightarrow Q \Rightarrow \boxed{\text{TRUE}} \)

**Analysis:** \( a \) is a rational number \( \Rightarrow a = \frac{p}{q}, \text{ and } \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}, q \neq 0 \)

\( b \) is a rational number \( \Rightarrow b = \frac{r}{s}, \text{ and } \frac{r}{s}, \text{ where } r, s \in \mathbb{Z}, s \neq 0 \)

\( a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \)

**Conclusion:** \( \boxed{a + b} \) is a rational number.

Thus, the sum of two rational numbers is rational.

---

6. Give a direct proof that if \( m \) and \( n \) are both perfect squares, then \( mn \) is also a perfect square.

**Solution:** Theorem: \( \forall (m(n) \rightarrow n(m)) \), \( m \) and \( n \) are both perfect squares.

**Proof:** DIRECT PROOF: \( P \Rightarrow Q \Rightarrow \boxed{\text{TRUE}} \)

**Hypothesis:** Assume \( P \Rightarrow Q \Rightarrow \boxed{\text{TRUE}} \)

**Analysis:** An integer \( \sqrt{a} \) is a perfect square if there is an integer \( b \) such that \( a = b^2 \).

\( m \) is a perfect square \( \Rightarrow m = b^2 \)

\( n \) is a perfect square \( \Rightarrow n = a^2 \)

**Conclusion:** \( mn = b^2 \cdot c^2 = (bc)^2 \)

Thus, if \( m \) and \( n \) are both perfect squares, then \( mn \) is also a perfect square.

---

HW: 1. Prove that if \( m+n \) and \( m+p \) are even integers, then \( m+n+p \) is even.

2. Prove that for all integers \( k \) and \( l \), if \( k+l \) are both odd, then \( k+l \) is even and \( k+l \) is odd.

---

**Type 2:** Problems based on indirect proof: CONTRAPOSITIVE.

\[ P \Rightarrow Q \Rightarrow \boxed{\text{TRUE}} \]

**Theorem:** \( \forall (P(n) \rightarrow Q(n)) \), \( P(n) \Rightarrow Q(n) \Rightarrow 70 \Rightarrow 71 \Rightarrow P \Rightarrow \boxed{\text{TRUE}} \)

**Proof:** If \( n \) is an integer and \( 3n+2 \) is odd, then \( n \) is odd.

**Solution:** Given: If \( n \) is an integer and \( 3n+2 \) is odd, then \( n \) is odd.

**Theorem:** \( \forall (P(n) \rightarrow Q(n)) \), \( P(n) \Rightarrow 3n+2 \text{ is odd} \Rightarrow n \text{ is odd} \Rightarrow Q(n) \Rightarrow \boxed{\text{TRUE}} \)
**Proof:** Indirect Proof by Contrapositive. \[ \neg p \Rightarrow \neg q \]

**Hypothesis:** \(70 \text{ is true}\) \(\Rightarrow\) \(q\) is an odd integer; \(70\) is an even integer.

**Analysis:** \(n\) is an even integer \(\Rightarrow\) \(n = 2k\), \(k\) is an integer.

\[ 3n + 2 = 3(2k) + 2 = 2(3k+1) = \text{Even int} \]

**Conclusion:** \(3n + 2 = 2(3k+1) = \text{Multiple of } 2\)
\[ \Rightarrow 7p = 3n + 2 \text{ is an even integer.} \]

**Hence:** \(70 \Rightarrow 7p \text{ is true} \Rightarrow p \Rightarrow q \text{ is true.}\)

---

2. Prove that if \(n\) is an integer and \(n^2\) is odd then \(n\) is odd.

**Solution:**

**Given:** If \(n\) is an integer and \(n^2\) is odd then \(n\) is odd.

**Theorem:** \(\forall n [\neg (\neg (\neg n)) \Rightarrow \neg \neg \neg (\neg n)]\), \(n^2\) is odd int \(\Rightarrow n\) is an int.

**Proof:** Indirect Proof by Contrapositive:

\[ 1 \Rightarrow \neg p \Rightarrow \neg q \Rightarrow \neg r \Rightarrow \neg s \Rightarrow \neg t \]

**Hypothesis:** Assume \(70\) is true \(\Rightarrow\) \(q\) is an odd integer.

\(70\) is an even integer.

**Analysis:** \(n\) is an even integer \(\Rightarrow n = 2k\), \(k\) is an integer.

\[ n^2 = (2k)^2 = 4k^2 = 2(2k^2) = \text{even int}, \text{ where } k = 2k^2. \]

**Conclusion:** \(n^2 = 2t = \text{Multiple of } 2\)
\[ \Rightarrow n^2 \text{ is an even integer.} \]
\[ \Rightarrow 7p = n^2 \text{ is an even integer.} \]
\[ \Rightarrow 70 \Rightarrow 7p \text{ is even.} \]

**If** \(n\) \(\text{is an integer and } n^2\) \(\text{is odd then } n\) \(\text{is odd.}\)

---

3. Show that if \(n\) is an integer and \(n^3 + 5\) is odd, then \(n\) is even.

**Solution:**

**Theorem:** \(\forall n [\neg (\neg (\neg n)) \Rightarrow \neg \neg \neg (\neg n)]\), \(n^3 + 5\) is odd int \(\Rightarrow n\) is an int.

**Proof:** Proof by Contrapositive:

\[ p \Rightarrow q \Rightarrow \neg p \text{ is even.} \]

**Hypothesis:** Assume \(70\) is true \(\Rightarrow\) \(q\) is an even integer.

\(70\) is an odd integer.

**Analysis:** \(n\) is odd \(\Rightarrow n = 2k + 1\), for some integer

\[ n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 1 + 2k^2 + 6k + 5. \]
\[ = (8k^3 + 1 + 2k^2 + 6k + 5) \]
\[ = 2(4k^3 + 6k^2 + 3k + 3) \]

**Conclusion:** \(n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3) = \text{Multiple of } 2\).

\[ \Rightarrow \neg n\text{ is an integer \& } \neg (n^3 + 5) \text{ is even.} \]
\[ \Rightarrow 70 \Rightarrow \neg 7p \Rightarrow p \text{ is even.} \]

**If** \(n\) \(\text{is an integer and } n^3 + 5\) \(\text{is odd then } n\) \(\text{is odd.}\)
Show that the statements are equivalent:

1. \( P_1 = n \) is an even integer; \( P_2 = (n-1) \) is an odd integer.
2. \( P_3 = n^2 \) is an even integer.

So: T.P. The statements are equivalent.

1. \( P_1 \Rightarrow P_2 \equiv T \) [Direct Proof]
2. \( P_2 \Rightarrow P_3 \equiv T \) [Direct Proof]
3. \( P_3 \Rightarrow P_1 \equiv T \) [Indirect Proof]

Hence, the statements are equivalent.
Prove that if \( n = ab \) where \( a, b \) are positive integers, then \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \).

**Solution:**

**Theorem:** \( n = \lceil \frac{p(n)}{q(n)} \rceil \), \( p(n) \# n = ab \) are the integer \( q(n) = a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \).

**Proof:** PROOF BY CONTRADICTION: \( \neg p \rightarrow \neg q \Rightarrow \neg p \rightarrow \neg q \)

**Hypothesis:** Assume \( \neg q \) is false \( \neg q \) \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \) is false.

**Analysis:** \( a > \sqrt{n} \) or \( b > \sqrt{n} \). Now \( ab > \sqrt{n} \sqrt{n} = n \).

- \( \neg q \Rightarrow ab > n \Rightarrow \sqrt{n} \leq ab \)

**Conclusion:** We observe that \( n < ab \) but \( n = ab \).

\( \Rightarrow \neg \) [Which is a contradiction to assumption].

Our Assumption is wrong.

In view of this contradiction, \( p \rightarrow q \).

---

Prove that \( \sqrt{2} \) is irrational by giving a proof of contradiction.

**Solution:**

**Theorem:** \( P: \sqrt{2} \) is irrational.

**Proof:** PROOF BY CONTRADICTION.

**Hypothesis:** Assume \( P \) is not true.

**Analysis:** \( \sqrt{2} \) is not irrational \( \Rightarrow \sqrt{2} \) is rational.

- \( \sqrt{2} = \frac{a}{b} \), there exist integers \( a \) and \( b \), where \( b \neq 0 \) and \( a \) and \( b \) have no common factors.

- \( 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2 \) is even \( \Rightarrow [a \) is even]

- \( 2b^2 = 4c^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2 \) is even \( \Rightarrow [b \) is even]

**Conclusion:** If \( a \) is even and \( b \) is even, \( a \) and \( b \) have factors 2.

\( \Rightarrow \neg \) [Which is contradiction to the statement that \( a \) and \( b \) have no common factors].

Our assumption is wrong.

\( \therefore \sqrt{2} \) is irrational.

---

**Type 4:** PROOF BY EQUIVALENCE

Prove the theorem "If \( n \) is a positive integer, then \( n \) is odd if and only if \( n^2 \) is odd.

**Solution:**

**Theorem:** If \( n \) is an even integer, then \( n \) is odd if and only if \( n^2 \) is odd.

**Symbolize:** \( \forall n [p(n) \iff q(n)] \); \( p(n) : n \) is odd

\( q(n) : n^2 \) is odd.

**Proof:** PROOF BY EQUIVALENCE: \( p \iff q \iff (p \iff q) \land (q \iff p) \)

**Step 1:** \( (p \iff q) \land (q \iff p) \)

**Step 2:** \( p \iff q \)

**Step 3:** \( q \iff p \)

**Step 4:** \( (p \iff q) \land (q \iff p) \)

**Step 5:** \( \iff \)

**Step 6:** \( \iff \)

**Step 7:** \( \iff \)

**Step 8:** \( \iff \)

**Step 9:** \( \iff \)

**Step 10:** \( \iff \)

**Step 11:** \( \iff \)

**Step 12:** \( \iff \)

**Step 13:** \( \iff \)

**Step 14:** \( \iff \)

**Step 15:** \( \iff \)

**Step 16:** \( \iff \)

**Step 17:** \( \iff \)

**Step 18:** \( \iff \)

**Step 19:** \( \iff \)

**Step 20:** \( \iff \)

**Step 21:** \( \iff \)

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**Step 23:** \( \iff \)

**Step 24:** \( \iff \)

**Step 25:** \( \iff \)

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**Step 27:** \( \iff \)

**Step 28:** \( \iff \)

**Step 29:** \( \iff \)

**Step 30:** \( \iff \)

**Step 31:** \( \iff \)

**Step 32:** \( \iff \)

**Step 33:** \( \iff \)

**Step 34:** \( \iff \)

**Step 35:** \( \iff \)

**Step 36:** \( \iff \)

**Step 37:** \( \iff \)

**Step 38:** \( \iff \)

**Step 39:** \( \iff \)

**Step 40:** \( \iff \)

**Step 41:** \( \iff \)

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**Step 46:** \( \iff \)

**Step 47:** \( \iff \)

**Step 48:** \( \iff \)

**Step 49:** \( \iff \)

**Step 50:** \( \iff \)

**Step 51:** \( \iff \)

**Step 52:** \( \iff \)

**Step 53:** \( \iff \)

**Step 54:** \( \iff \)
1. Find the truth table for the statement $P \Rightarrow q$.

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<th>$P$</th>
<th>$q$</th>
<th>$P \Rightarrow q$</th>
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</table>

2. Construct a truth table for $(P \Rightarrow q) \Rightarrow (q \Rightarrow P)$.

<table>
<thead>
<tr>
<th>$P$</th>
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<th>$q \Rightarrow P$</th>
<th>$P \Rightarrow q$</th>
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3. Construct the truth table for $(P \Rightarrow q) \iff (\neg P \Rightarrow \neg q)$.

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<tr>
<th>$P$</th>
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<th>$\neg P \Rightarrow \neg q$</th>
<th>$(P \Rightarrow q) \iff (\neg P \Rightarrow \neg q)$</th>
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</table>

4. Using the truth table verify $(P \land q) \land T (P \lor q)$.

<table>
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<tr>
<th>$P$</th>
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5. Give the truth value of $T \iff \neg T \land F$

$T \iff (\neg T \land F) \equiv T \iff F \equiv F$

6. How many rows are needed for the truth table of the formula $(P \land q) \iff (\neg q \lor r) \Rightarrow T$.

Sol: The given statement formula involves 5 variables namely $P, q, r, s, t$. The table contains $2^5 = 32$ rows.

7. State the truth value of "if tigers have wings then the earth travels around the sun".

Sol: Propositions: $P$: Tigers have wings
$Q$: Earth travels around the sun

Symbolic form: $P \Rightarrow Q$

Truth value: Refer Problem No.1.
Write the symbolic representation of "If it rains today, then I buy an umbrella."

Sol: Propositions: 

\[ P: \text{It rains today} \]
\[ Q: \text{I buy an umbrella} \]

Symbolic form: \[ P \rightarrow Q \]

Express in symbolic form "The crop will be destroyed if there is a flood."

Sol: Restatement: If there is a flood then the crop will be destroyed.

Proposition: 

\[ P: \text{There is a flood} \]
\[ Q: \text{Crop will be destroyed} \]

Symbolic form: \[ P \rightarrow Q \]

Express in symbolic form "Good food is not cheap."

Sol: Restatement: If there is good food then it is not cheap.

Proposition: 

\[ P: \text{There is good food} \]
\[ Q: \text{It is cheap} \]

Symbolic form: \[ P \rightarrow \neg Q \]

Is the statement "5+2 iff 5-2 > 0" true?

Sol: \[ P: 5 \geq 2 \]
\[ Q: 5-2 > 0 \]

\[ P \leftrightarrow Q \text{ is true} \]

Is the statement "Chennai is in Russia iff 3+6 = 10" true?

Sol: \[ P: \text{Chennai is in Russia} \]
\[ Q: 3+6 = 10 \]
\[ P \leftrightarrow Q \text{ is true} \]

Let \( P: \text{I will study Discrete Mathematics} \)
\( Q: \text{I will watch TV} \)
\( R: \text{I am in a good mood} \)

Write the following statement using \( P \), \( Q \), \( R \) and logical connectives

(1) If I donot study Discrete and I watch TV, then I am in good mood.
(2) If I am not in a good mood, then I will not watch TV or I will study Discrete Mathematics.
(3) If I am in a good mood, then I will study discrete Mathematics or I will watch TV.
(4) I will watch TV and I will not study Discrete Mathematics iff I am in a good mood.

Sol: Propositions:

\[ P: \text{I will study Discrete mathematics} \]
\[ Q: \text{I will watch TV} \]
\[ R: \text{I am in a good mood} \]

Symbolic form:

1. \[ \neg P \lor Q \rightarrow R \]
2. \[ Q \lor \neg Q \rightarrow P \]
3. \[ \neg P \rightarrow Q \]
4. \[ R \rightarrow P \]
5. \[ q \lor \neg q \rightarrow y \]
Write down negation of the following proposition:
(a) The summer in Chennai is hot and sunny.
Sol: The summer in Chennai is not hot or not sunny.
(b) Some people have no two wheeler.
Sol: Every person has a two wheeler.
(c) Every even integer greater than 2 is a sum of two primes.
Sol: There is at least one even integer greater than 2 that is not the sum of two primes.

15. What are the contrapositive, the converse and the inverse of the conditional statement of the following?
(a) If you work hard then you will be rewarded. [MLB 113]
Sol: Proposition: P: You work hard
q: You will be rewarded.

Given Implication: P \rightarrow q.
Contrapositive: \neg q \rightarrow \neg p \Rightarrow If you will not be rewarded, then you will not work.
Converse: q \rightarrow p \Rightarrow If you will be rewarded, then you work hard.
Inverse: \neg p \rightarrow \neg q \Rightarrow If you will not work hard, then you will not be rewarded.

(b) If it is raining, then I get wet.
Sol: Proposition: P: It is raining.
a: I get wet.

Given Implication: P \rightarrow a
Contrapositive: \neg a \rightarrow \neg p \Rightarrow If I do not get wet, then it is not raining.
Converse: a \rightarrow p \Rightarrow If I get wet, then it is raining.
Inverse: \neg p \rightarrow \neg a \Rightarrow If it is not raining, then I will not get wet.

16. Find the contrapositive of the inverse of P \rightarrow q; [AML 08]
Sol: Given Implication: P \rightarrow q; Inverse of P \rightarrow q: \neg p \rightarrow \neg q
Contrapositive of inverse: \neg q \rightarrow \neg p.

17. Define Tautology, Contradiction, Contingency with Example.
Sol: Tautology: A statement formula which is true always irrespective of the truth values of the individual variable is called Tautology.
Ex: P \lor \neg P \equiv T
Contradiction: A statement formula which is always false.
Ex: P \land \neg P \equiv F
Contingency: A statement formula which is neither Tautology nor Contradiction.
Ex: P \rightarrow \phi is a contingency.
8. Is \((P \land (P \lor q)) \rightarrow q\) a Tautology?  

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>\sim P</th>
<th>P \lor q</th>
<th>T \land (P \lor q)</th>
<th>\sim P \land (P \lor q)</th>
<th>A \rightarrow q</th>
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<tbody>
<tr>
<td>T</td>
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</table>

9. Show that \((P \rightarrow (Q \rightarrow R)) \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)]\) is a Tautology.  

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>Q \rightarrow R</th>
<th>P \rightarrow (Q \rightarrow R)</th>
<th>P \rightarrow Q</th>
<th>P \rightarrow R</th>
<th>B \rightarrow C</th>
<th>A \rightarrow D</th>
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</table>

10. Check the following proposition is Tautology \([CP \rightarrow Q) \rightarrow R]\) v \(\sim P\)  

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>P \rightarrow Q</th>
<th>(P \rightarrow Q) \rightarrow R</th>
<th>\sim P</th>
<th>[(P \rightarrow Q) \rightarrow R] \lor \sim P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

All the entries in the resulting column is not T, hence the given proposition is not a Tautology.

11. When do you say that two Compound propositions are equivalent?

Sol: \(P \Leftrightarrow Q\) when \(P \Leftrightarrow Q\) is a Tautology.

12. Show that proposition \((P \implies Q) \& \sim P \lor Q\) are logically equivalent.

Sol: \(T : P \rightarrow Q\) = \((\sim P \lor Q)\) [Using Truth Table]

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>\sim P</th>
<th>P \rightarrow Q</th>
<th>\sim P \lor Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
Using Truth Table, show that the proposition \( P \land (P \lor Q) \) is a Tautology.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \lor Q</th>
<th>( T(P \lor Q) )</th>
<th>( P \land (P \lor Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

2. Show that \( (P \rightarrow Q) \land (Q \rightarrow R) \) is logically equivalent to \( (P \lor Q) \rightarrow R \).

Sol: T.P \( [(P \rightarrow Q) \land (Q \rightarrow R)] \iff (P \lor Q) \rightarrow R \)

L.H.S \[= [T(P \rightarrow Q) \land (Q \rightarrow R)] \]

\[= [T(P \lor Q)] \land [T(Q \lor R)] \]

\[= T \land T \]

\[= T \]

R.H.S

```

3. Show that \( P \land (Q \lor R) \land (Q \land R) \) is a Tautology.

Sol: T.P \( P \iff Q \land R \) [Use Truth Table]

<table>
<thead>
<tr>
<th>P</th>
<th>TP</th>
<th>TTP</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

\[ \therefore \text{Truth value of } P = \text{Truth value of } TTP \]

\[ \therefore P \iff TTP \]

4. Show that \( T \land (T \lor R) \land (Q \lor R) \land (P \land Q) \) is logically equivalent to \( R \).

Sol: L.H.S \[= [T \land (T \lor R) \land (Q \lor R) \land (P \land Q)] \]

\[= [T \land (T \lor R)] \land [T \lor (R \land Q) \land (P \land Q)] \]

\[= T \land (T \lor R) \land (Q \lor R) \land (P \land Q) \]

\[= T \land R \land (Q \lor R) \land (P \land Q) \]

\[= T \land R \]

\[= R \]

R.H.S

```

5. Show that \( P \rightarrow (Q \rightarrow R) \equiv (P \land Q) \rightarrow R \equiv P \rightarrow (Q \lor R) \equiv 2 \) is logically equivalent.

Sol: L.H.S: \[= P \rightarrow (Q \rightarrow R) \equiv (P \land Q) \rightarrow R \equiv P \rightarrow (Q \lor R) \equiv 2 \]

\[= T \land (T \lor R) \]

\[= T \land R \land (Q \lor R) \land (P \land Q) \]

\[= T \land (P \land Q) \land (Q \lor R) \land (P \land Q) \]

\[= T \land (P \land Q) \land (Q \lor R) \land (P \land Q) \]

\[= (P \land Q) \rightarrow R \equiv 2 \]

Condition Law.
23. Show that \( \neg(\neg p \land q) \rightarrow (\neg p \lor (\neg q \lor r)) \iff \neg p \lor q \land r \)

Sol: L.H.S \(\equiv \neg(\neg p \land q) \rightarrow (\neg p \lor (\neg q \lor r)) \)

\[\equiv \neg (\neg p) \lor (\neg q) \lor r \]
\[\equiv p \lor \neg q \lor r\]

Condition Law

\[\equiv \neg p \lor \neg q \lor r\]

Double negation

\[\equiv \neg p \lor \neg q \lor r\]

Associative Law

\[\equiv \neg p \lor \neg q \lor r\]

Idempotent Law

\[\equiv \neg p \lor \neg q \lor r\]

Commutative Law

\[\equiv \neg p \lor \neg q \lor r\]

Distributive Law

\[\equiv \neg p \lor \neg q \lor r\]

Dominance Law

\[\equiv \neg p \lor \neg q \lor r\]

Commutative Law

\[\equiv \neg p \lor \neg q \lor r\]

Identity Law

Given statement formulae is logically equivalent.

24. Show that \( p \rightarrow (q \lor r) \iff (p \rightarrow q) \land (p \rightarrow r) \)

Sol: R.H.S \(\equiv (p \rightarrow q) \land (p \rightarrow r) \)

\[\equiv (\neg p \lor q) \land (\neg p \lor r) \]

Condition Law

\[\equiv (\neg p \lor q) \land (\neg p \lor r) \]

Associative Law

\[\equiv \neg p \lor (q \land r) \]

Idempotent Law

\[\equiv \neg p \lor (q \land r) \]

Commutative Law

\[\equiv \neg p \lor (q \land r) \]

Distributive Law

\[\equiv \neg p \lor (q \land r) \]

Dominance Law

\[\equiv \neg p \lor (q \land r) \]

Commutative Law

\[\equiv \neg p \lor (q \land r) \]

Identity Law

Given statement formulae is logically equivalent.

25. Write the equivalent formula for \( p \land (q \leftrightarrow r) \)

Sol: \( p \land (q \leftrightarrow r) \equiv p \land [(q \rightarrow r) \land (r \rightarrow q)] \)

Condition Law

\[\equiv p \land [(\neg q \lor r) \land (\neg r \lor q)] \]

Condition Law

\[\equiv p \land [(\neg q \land \neg r) \lor (r \land q)] \]

Commutative Law

\[\equiv p \land [(\neg q \land \neg r) \lor (r \land q)] \]

Distributive Law

\[\equiv p \land [(\neg q \land \neg r) \lor (r \land q)] \]

Dominance Law

\[\equiv p \land [(\neg q \land \neg r) \lor (r \land q)] \]

Commutative Law

\[\equiv p \land [(\neg q \land \neg r) \lor (r \land q)] \]

Identity Law

Given statement formulae is equivalent formula.
Define Tautological Implication. [AIM '10]
Sol: A statement formula A is said to be tautologically imply a statement formula B iff A → B is a tautology.

Prove the following implication P ⊃ (P V Q).
Sol: Let P ⊃ (P V Q) is a Tautology.
P ⊃ (P V Q) ≡ T P V (P V Q) [Condition Law]
≡ (T P V P) V Q [Associative Law]
≡ T V Q [Complementary Law]
≡ Q V T [Commutative Law]
≡ T [Dominance Law]

Define Dual of a Statement formula. [AIM '09]
Sol: Two formulas A & A* are said to be duals of each other if either one can be obtained from other by replacing
V by V , ^ by ^ , T by F & F by T.

Eg (a) Dual [(P V Q) V T] = (P V Q) AF
(b) Dual [P ^ (Q V T)] = P V (Q ^ T)
(c) Dual [(P V F) ^ (Q V T)] = (P AF) ^ (Q V T)

Define DNF
Sol: Disjunctive Normal Form (DNF) is sum of Elementary Product.
1e (Elementary Product) V (Elementary Product) V ... V (Elementary Product)

Define CNF
Sol: Conjunction Normal Form (CNF) is product of Elementary Sum.
1e (Elementary Sum) ^ (Elementary Sum) ^ ... ^ (Elementary Sum)

Obtain DNF of P ^ (P ⊃ Q)
Sol: P ^ (P ⊃ Q) ≡ P ^ (T P Q) [Condition Law]
≡ P ^ (P ^ T P ^ Q) [Distributive Law]
≡ Sum of Elementary Product
≡ DNF.

Obtain Conjunctive Normal Form of P ^ (P ⊃ Q) [A-U '08]
Sol: P ^ (P ⊃ Q) ≡ P ^ (T P Q)
≡ Product of Elementary Sum
≡ CNF.

What are the possible Truth values for an atomic structures.
Sol: The possible Truth values of atomic statements are TRUE or FALSE.
Find the conjunctive normal form of \((\overline{Q \vee (P \wedge R)}) \wedge (\overline{C \vee (P \wedge R)})\).

**Sol:**
\[
(\overline{Q \vee (P \wedge R)}) \wedge (\overline{C \vee (P \wedge R)})
\]
\[
= (\overline{Q \vee (P \wedge R)}) \wedge (\overline{C \vee (P \wedge R)})
\]
\[
= (\overline{Q \vee (P \wedge R)}) \wedge (\overline{C \vee (P \wedge R)})
\]
\[
= \text{Product of Elementary Sum}
\]
\[
= \text{CNF}
\]

Write an equivalent formula for \(P \wedge (Q \Rightarrow R)\) which contains neither the biconditional nor the conditional.

**Sol:**
\[
P \wedge (Q \Rightarrow R)
\]
\[
= P \wedge (\overline{Q} \vee (Q \Rightarrow R))
\]
\[
= P \wedge (\overline{Q} \vee (Q \Rightarrow \overline{Q}))
\]
\[
= P \wedge (\overline{Q} \vee \overline{Q})
\]
\[
= \text{Equivalent Formula}
\]

Obtain principal disjunctive normal forms of \(TPVA\).

**Sol:**
\[
TPVA = (TP \wedge R) \vee (\overline{Q} \wedge A \wedge T)
\]
\[
= (TP \wedge (\overline{Q} \wedge A \wedge T)) \vee (\overline{Q} \wedge (P \wedge T))
\]
\[
= (TP \wedge \overline{Q}) \vee (TP \wedge A \wedge T) \vee (\overline{Q} \wedge P \wedge T)
\]
\[
= TP \wedge A \wedge T \vee \overline{Q} \wedge P \wedge T
\]
\[
= \text{Sum of Minterms}
\]
\[
= \text{PDNF}
\]

Define a rule of universal specification.

**Sol:**
\[
\forall x \ A(x) \Rightarrow A(c)
\]

When a set of formulae is consistent and inconsistent.

**Sol:**

**Consistent:** A set of formulae \(H_1, H_2, \ldots, H_m\) is said to be consistent if their conjunction implies tautology.

**Inconsistent:** A set of premises \(H_1, H_2, \ldots, H_m\) is said to be inconsistent if their disjunction implies contradiction.

Define 4-place predicate.

**Sol:** If there are 4 names of objects associated with a predicate, then it is known as 4-place predicate.

Show that \(T(P \wedge A)\) follows from \(TPA70\).

**Sol:**
Premises: \(TPA70\)
Conclusion: \(T(P \wedge A)\)

Method: Indirect Method.
Additional Premises: \(T(\overline{C\text{Conc}(T(PA \wedge A))}) = P \wedge A\).
### Exercise 1

Determine whether C follows logically from the premises:

H1: P → Q; H2: P; C: Q.

**Steps**

1. **Premises**: P → Q, P
   **Reason**: Additional Premises.

2. **Premises**: P
   **Reason**: Rule T: Simplification.

3. **Premises**: Q
   **Reason**: Rule T: Simplification.

4. **Premises**: TP ∧ Q
   **Reason**: Rule P.

5. **Premises**: TP
   **Reason**: Rule T: Simplification.

6. **Premises**: TP ∧ P
   **Reason**: 2, 5: Rule T: Conjunction.

7. **Premises**: TP, Rule T: P ∧ T = F

**Conclusion**: Q

### Exercise 2

**If this number is divisible by 4, then it is divisible by 2.**

**This number is not divisible by 2. Therefore this number is not divisible by 4.**

**Solution:** Let P: This number is divisible by 4.
Q: It is divisible by 2.

**Symbolic Form:** P → Q, .

The given argument may be written as **P → Q.**

**:. By Modus Tollens the argument is valid.**

### Exercise 3

**Test the validity of the argument.**

If it rains today, then I will not go for a movie today.
If I do not go for a movie today, then I will go for the movie tomorrow. Therefore, if it rains today, then I will go for the movie tomorrow.

**Steps:**

1. **Propositions:** P: It is raining today.
   Q: I will not go for movie today.
   R: I will go for a movie tomorrow.

2. **Symbolic Form:** P → Q, Q → R

The given argument can be written as **P → Q.**

**:. By Hypothetical Syllogism the argument is valid.**
If the premises $P \land Q \land R$ are inconsistent, $P \land Q \land R$ is a contradiction from $P \land Q$.

**Sol:** Given: $P, Q, R$ are inconsistent. \( \therefore P \land Q \land R \equiv F \)

- **Prop:** $P \land Q \Rightarrow TR$
  - Suppose $P \land Q \Rightarrow TR$ is false.
  - $(P \land Q) \Rightarrow TR \equiv F$
  - $T(P \land Q) \lor TR \equiv F$
  - $T(P \land Q \land R) \equiv F$
  - $F \equiv EF$, is a contradiction.

- **Sol:** Prove that $P \land Q$, $Q \Rightarrow R \Rightarrow P$

**Premises:** $P \land Q$, $Q \Rightarrow R$

**Conclusion:** $R$

<table>
<thead>
<tr>
<th>Steps</th>
<th>Premises</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P$</td>
<td>Rule P</td>
</tr>
<tr>
<td>2</td>
<td>$P \Rightarrow Q$</td>
<td>Rule P</td>
</tr>
<tr>
<td>3</td>
<td>$Q$</td>
<td>$\Rightarrow {2,1,3} \Rightarrow Rule T: Modus Ponens $</td>
</tr>
<tr>
<td>4</td>
<td>$Q \Rightarrow R$</td>
<td>Rule P</td>
</tr>
<tr>
<td>5</td>
<td>$R$</td>
<td>$\Rightarrow {4,3} \Rightarrow Rule T: Modus Ponens $</td>
</tr>
</tbody>
</table>

Express the statement, "Some people who trust others are rewarded" in symbolic form.

**Sol:** Propositions:
- $P(x)$: $x$ is a person
- $Q(x)$: $x$ trusts others
- $R(x)$: $x$ is rewarded

**Symbolic form:** $(\exists x)(P(x) \land Q(x) \land R(x))$

Symbolise the expression "For any $x$ and any $y$, if $x$ is taller than $y$, then $y$ is not taller than $x$".

**Sol:** Let $T$ be the predicate 'is taller than'.
- $T(x,y)$: $x$ is taller than $y$
- $T(y,x)$: $y$ is not taller than $x$. (Note: The given statement function $\forall x \forall y [T(x,y) \Rightarrow T(y,x)]$)

Write in symbolic form "Every student in this examination hall knows C programming or Java".

**Sol:** Universe of discourse: Set of all students
- Propositions:
  - $A(x)$: $x$ is in this examination hall
  - $B(x)$: $x$ knows C programming
  - $C(x)$: $x$ knows Java

**Symbolic form:** \( \forall x [A(x) \rightarrow (B(x) \lor C(x)) ] \)
Express using quantifiers, predicates & logical connectives, the following statement.

(a) Everyone is your friend and is perfect.
(b) Everyone in your class has a cellular phone.
(c) Someone in your class does not play cricket.

Sol: Universe of Discourse: Set of students in your class.

Propositions:
- \( p(x) \): x is your friend.
- \( q(x) \): x is perfect.
- \( r(x) \): x has cellular phone.
- \( h(x) \): x plays Hockey.

Symbolic forms:
- (a) \( \forall x (p(x) \land q(x)) \)
- (b) \( \forall x r(x) \)
- (c) \( \exists x \neg h(x) \)

56. Translate into English, \( \forall x \forall y (x > 0) \land (y < 0) \rightarrow (xy < 0) \)

Sol: The product of a positive real no. and a negative real no. is a negative real number.

57. Translate the statement \( \forall x [\forall f(x) \lor \exists y (f(x) = f(xy))] \)
into English, where \( f(x) \): x has a computer, \( F(x,y) \): x & y are friends and the universe of discourse is the set of all students in your college.

Sol: Every student in your college has a computer or has a friend who has a computer.

58. What are the negations of the statements \( \forall x \exists y (xy = 1) \)
so that no negation proceeds a quantifier?

Sol: \( \exists x \forall y (xy \neq 1) \)

59. What are the negations of the statements \( \forall x (x^2 - x) \)
and \( \exists x (x^2 = 2) \)?

Sol: \( \neg p(x) = (\exists x) \neg (x^2 - x) \); \( \neg q(x) = (\forall x) (x^2 \leq x) \)

60. Write the negation of the statement \( (\exists x) (\forall y) p(x,y) \)

Sol: \( (\forall x) \neg (\forall y) p(x,y) \).

61. Let \( A(x) \) denote the propositional function \( 'x is less than 5' \).

(a) Find the truth value statement \( \forall x A(x), \exists x A(x) \) if the universe of discourse is \( \{ -1, 0, 1, 2, 4, 3 \} \).

Sol: \( A = \{ -1, 0, 1, 2, 4, 3 \} \); \( \forall x A(x) \) & \( \exists x A(x) \)

We have \( \forall x A(x) \) is true and \( \exists x A(x) \) is true.
Given \( P = \{3, 4, 5, 6, 7\} \) state the truth value of the statement 
\((\exists x \in P) (x + 3 = 10)\)

**Solution:**
- Universe of Discourse: \( P = \{3, 4, 5, 6, 7\} \)
- **Statement:** \( P(x) : x + 3 = 10 \)
- **Truth value of:** FALSE [There is no \( x \) in \( P \) for which \( x + 3 = 10 \)]

Let \( E = \{-1, 0, 1, 2\} \) denote the universe of discourse. If \( P(x, y) : x + y = 1 \), find the truth value of \( \exists x \forall y \ P(x, y) \).

**Solution:**
- Universe of discourse: \( E = \{-1, 0, 1, 2\} \)
- **Statement:** \( P(x, y) : x + y = 1 \)
- **Truth value of:** FALSE [There is no one \( y \) that works for all \( y \).

Write the truth value of \( (\exists x) P(x) \) where \( P(x) : x^2 > 10 \) with \( U = \{1, 2, 3, 4\} \).

**Solution:**
- **Statement:** \( P(x) : x^2 > 10 \)
- **Truth value of:** \((\exists x) P(x)\): TRUE [\( x = 4 \), \( x^2 > 10 \)]

What are the truth values of following quantifications over the set of real numbers.

**Solution:**
1. \((\forall x) P(x) \) where \( P(x) = (x + 1) > x \): TRUE
2. \((\exists x) Q(x) \) where \( Q(x) = x = x + 1 \): FALSE

If \( P(x, y) \) be the statement \( x + y = y + x \). What is the truth value of the quantified statement \( (\forall x) (\forall y) P(x, y) \)?

**Solution:**
- **Statement:** \( P(x, y) \): \( x + y = y + x \)
- **Truth value of:** \((\forall x)(\forall y) P(x, y)\): TRUE.

Let \( Q(x) \) denote \( x + y = 0 \). What are the truth values of the quantified statements \((\exists y) (\forall x) Q(x, y) \) & \((\forall x) (\exists y) Q(x, y) \)?

**Solution:**
- **Statement:** \( P(x, y) \): \( x + y = 0 \)

1. Truth value of \((\exists y)(\forall x) Q(x, y)\): FALSE
2. Truth value of \((\forall x)(\exists y) Q(x, y)\): TRUE \( \Rightarrow \) [For all real no: \( x \) there exists a real no: \( y \) s.t. \( x + y = 0 \)]

Let \( Q(x, y, z) \) be the statement \( x + y = z \). What are the truth values of \( (\forall x) (\forall y) (\exists z) Q(x, y, z) \) & \( (\exists z) (\forall x) (\forall y) Q(x, y, z) \)?

**Solution:**
- **Statement:** \( P(x, y) \): \( x + y = z \)

1. Truth value of \((\forall x)(\forall y)(\exists z) Q(x, y, z)\): TRUE
2. Truth value of \((\exists z)(\forall x)(\forall y) Q(x, y, z)\): FALSE
What is the truth value of $(\exists x)(pcx \Rightarrow (\forall x)) \land T$ where $pcx : x > 2$ and $\forall x: x = 0$ & $T$ is any Tautology. Universe of discourse is $\mathbb{N}$.

**Solution:**
- Universe of Discourse: $\mathbb{N}$
- Statement: $pcx: x > 2$, $\forall x: x = 0$
- Truth value of $(\exists x)(pcx \Rightarrow (\forall x)) \land T$: TRUE
- $x$ is allowed to take only value $x = 1$.
- $pc(1): FALSE$
- $\forall(1): FALSE$
- $pcx \Rightarrow (\forall x)$ is TRUE
- $[pcx \Rightarrow (\forall x)] \land T$ is TRUE.

Over the set of real numbers, what is the truth values of $(\forall x)(\exists y)(xy = 1)$

**Solution:**
- Universe of discourse: Set of real numbers
- Statement: $xy = 1$
- Truth value of $(\forall x)(\exists y)(xy = 1)$: TRUE
- For every real no. $x$ there is a real no. $y$ s.t. $xy = 1$.

Define free and bounded variables.

**Solution:**
1. **Bounded Variable**: The variable is said to be bound if it is concerned with either universal $(\forall x)$ or existential $(\exists x)$ quantifier.
2. **Scope**: The scope of the quantifier is the formula immediately following the quantifier.
3. The variable which is not concerned with any quantifier is called free variable.

Write the scope of the quantifiers in the formula:
- (a) $\forall x (pcx \Rightarrow \exists y R(x, y))$
- (b) $\exists x pcx \land (\forall x)$

**Solution:**
- (a) Formula: $\forall x [pcx \Rightarrow \exists y R(x, y)]$
  - Quantifiers: $\forall x$, $\exists y$
  - Scope of $\forall x$: $pcx \Rightarrow \exists y R(x, y)$
    - $\exists y: R(x, y)$
- (b) Formula: $\exists x pcx \land (\forall x)$
  - Quantifiers: $\exists x$
  - Scope of $\exists x$: $pcx$
    - $x$ is bound in $pcx$
    - $x$ is free in $(\forall x)$. 
Given an indirect proof of the theorem "If \(3n+2\) is odd, then \(n\) is odd.

**Indirect Method:**
Assume the conclusion is false and come to a contradiction.
1. Assume \(n\) is even
   \[ n = 2k \quad k \text{ is an integer} \]
Then \(3n+2 = 3(2k) + 2 = 2(3k+1)\)
   \[ 3n+2 \text{ is even} \]
   \[ \Rightarrow \Rightarrow \]
   \[ \therefore \text{Our assumption is wrong} \]
Hence \(n\) is odd and the given implication is true.

Let \(n\) be an integer. Prove that if \(n^2\) is odd, then \(n\) is odd.

**Sol:** Here, the conditional to be proved is \(p \Rightarrow q\), where
- \(p: n^2\) is odd
- \(q: n\) is odd

We first prove that the contrapositive \(\neg q \Rightarrow \neg p\) is true.
Assume that \(\neg q\) is true. Assume that \(n\) is not an odd integer.
Then \(n = 2k\), where \(k\) is an integer.
Consequently, \(n^2 = (2k)^2 = 4k^2 = \text{Even integer}\).
   \[ \Rightarrow n^2 \text{ is not odd} \]
   \[ \Rightarrow p \text{ is false (by TP is true)} \]
This proves the contrapositive statement \(\neg q \Rightarrow \neg p\).
The proof of the contrapositive \(\neg q \Rightarrow \neg p\) serves as an indirect proof of the given statements \(p \Rightarrow q\).
1. SIMPLIFICATION BY TRUTH TABLE AND WITHOUT TRUTH TABLE

1. Show that \( p \lor (q \land r) \) and \( (p \lor q) \land (p \lor r) \) are logically equivalent. \[ N/D \ 2010 \]

2. Without using the truth table, prove that \( \sim p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \lor r) \). \[ N/D \ 2010 \]

3. Prove \( ((p \lor q) \land \sim (p \land (q \lor r))) \lor (\sim p \lor q) \lor (\sim p \land q) \lor (p \rightarrow r) \) is a Tautology. \[ N/D \ 2013, A/M \ 2015 \]

4. Prove that \( (P \rightarrow Q) \land (R \rightarrow Q) \Rightarrow (P \lor R) \rightarrow Q \). \[ M/J \ 2013 \]

5. Prove that \( (p \rightarrow q) \land (q \rightarrow r) \Rightarrow (p \rightarrow r) \). \[ M/J \ 2014 \]

2. PCNF AND PDNF

1. Without using truth table find the PCNF and PDNF of \( p \rightarrow (q \land p) \land (\sim p \rightarrow (q \land \sim r)) \). \[ A/M \ 2011 \]

2. Find the Principal Disjunctive normal form of the statement \( (q \lor (p \land r)) \land \sim ((p \lor r) \land q) \). \[ N/D \ 2012 \]

3. Obtain the PDNF and PCNF of the statement \( p \lor (\sim p \rightarrow (q \lor (q \rightarrow r))) \). \[ N/D \ 2010 \]

4. Show that \( (\sim p \rightarrow R) \land (Q \leftrightarrow P) \equiv (P \lor Q \lor R) \land (P \lor Q \lor R) \land (P \lor Q \lor R) \land (P \lor Q \lor R) \land (P \lor Q \lor R) \). \[ M/J \ 2013 \]

3. THEORY OF INFERENCE

1. Show that \( (P \rightarrow Q) \land (R \rightarrow S), (Q \land M) \land (S \rightarrow N), \sim (M \land N) \) and \( (P \rightarrow R) \Rightarrow P \land P \rightarrow R \Rightarrow P \). \[ A/M \ 2011 \]

2. Show that \( (p \rightarrow q) \land (r \rightarrow s), (q \rightarrow t) \land (s \rightarrow u), \sim (l \land u) \) and \( (P \rightarrow R) \Rightarrow P \). \[ A/M \ 2015 \]

3. Prove that the following argument is valid: \( p \rightarrow q, r \rightarrow q, r \Rightarrow p \). \[ M/J \ 2012 \]

4. Prove that the premises \( a \rightarrow (b \rightarrow c), d \rightarrow (b \land c) \) and \( (a \land d) \) are inconsistent. \[ N/D \ 2010 \]

5. Using indirect method of proof, derive \( p \rightarrow s \) from the premises \( p \rightarrow (q \land r), q \rightarrow p, s \rightarrow p \) and \( p \). \[ N/D \ 2011 \]

Prove that \( A \rightarrow D \) is a conclusion from the premises \( A \rightarrow B \lor C, B \rightarrow \sim C \) and \( D \rightarrow C \) by using conditional proof. \[ M/J \ 2011 \]
7. Show that the hypothesis, "It is not sunny this afternoon and it is colder than yesterday," "Will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip" and "If we take a canoe trip, then we will be home by sunset" leads to the conclusion "we will be home by sunset." [N/D 2012][N/D 2013]

8. Determine the validity of the following argument: If 7 is less than 4, then 7 is not a prime number, 7 is not less than 4. Therefore 7 is a prime number. [M/J 2012]

9. State and explain the proof methods [N/D 2013]

10. Prove that \( \sqrt{2} \) is irrational by giving a proof using contradiction. [N/D 2011][M/J 2013][N/D 2013]

4. **Quantifiers**

1. Show that \((\forall x)(p(x) \rightarrow q(x)), (\exists y)p(y) \Rightarrow (\exists x)q(x)\). [M/J 2012]

2. Use the indirect method to prove that \((\forall x)(p(x) \lor q(x)) \Rightarrow (\forall x)p(x) \lor (\exists x)q(x)\). [A/M 2011][N/D 2011][A/M 2011]

3. Use the indirect method to prove that the conclusion \(\exists xQ(x)\) follows from the premises \(\forall x[p(x) \rightarrow q(x)]\) and \(\exists y p(y)\). [N/D 2012]

4. Prove that \((\forall x)(p(x) \lor q(x)) \Rightarrow (\forall x)p(x) \lor (\exists x)q(x)\). [M/J 2013]

5. Prove that \(\forall x(p(x) \rightarrow q(x)), \forall x(r(x) \rightarrow \neg q(x)) \Rightarrow \forall x(r(x) \rightarrow \neg p(x))\). [N/D 2010]

6. Show that \((\exists x)(p(x) \land q(x)) \Rightarrow (\exists x)p(x) \land (\exists x)q(x)\). Is the converse true. [N/D 2013]

7. Show that \((\exists x)p(x) \rightarrow \forall x q(x) \Rightarrow \forall x(p(x) \rightarrow q(x))\). [M/J 2014]

8. Show that the statement "Every positive integer is the sum of the squares of three integers" is false. [N/D 2011]

9. Verify the validity of the following argument "Every living thing is a plant (or) an animal. John’s gold fish is alive and it is not a plant. All animals have hearts. Therefore John’s gold fish has a heart". [N/D 2012]

10. Write the symbolic form and negate the following statement:

(i) Every one who is healthy can do all kinds of work.

(ii) Some people are not admired by everyone.

(iii) Every one should help his neighbors, or his neighbors will help him.

(iv) Everyone agrees with someone and someone agrees with everyone. [A/M 2015]

11. Verify that validating of the following inference, "If one person is more successful than another, then he has worked harder to deserve success. Ram has not worked harder than siva. Therefore, Ram is not more successful than siva. [A/M 2011]
ASSIGNMENT 1
MA6566- DISCRETE MATHEMATICS
UNIT -1

1. \( \neg(p \leftrightarrow q) \iff (p \land \neg q) \lor (\neg p \land q) \) using truth table and law.

2. Write the statement in symbolic form
   (a) Some integers are not square of any integers.
   (b) The crop will be destroyed if there is a flood.
   (c) Annu can access the internet from campus only if she is a computer science major or she is not a Freshgirl.

3. Give the converse, contrapositive and inverse of the following implication
   (a) If it is raining, then I get wet
   (b) If Raja is a poet, then he is poor

4. Show that \( (p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \) is a tautology.

5. \( (p \land q) \rightarrow r \iff (p \rightarrow r) \lor (q \rightarrow r) \)

6. \( (q \rightarrow (p \land \neg p)) \rightarrow (r \rightarrow (p \land \neg p)) \Rightarrow (r \rightarrow q) \)

7. Obtain the PCNF and PDNF of \( (p \rightarrow (q \land r)) \land (\neg p \rightarrow (\neg q \land \neg r)) \). Using truth table and law.

8. \( R \land S \) can be derived from \( P \rightarrow Q, Q \rightarrow \neg R, R, P \lor (R \land S) \)

9. Use indirect method and prove that \( p \rightarrow q, r \rightarrow q, s \rightarrow (p \lor r), s \Rightarrow q \).

10. Derive using conditional proof \( P, P \rightarrow (Q \rightarrow R \land S) \Rightarrow Q \rightarrow S \)

11. Show that the following premises are inconsistent:
    A diagnostic message is stored in a buffer or it is retransmitted. A diagnostic message is not stored in the buffers. If a diagnostic message is stored in buffer then it is retransmitted. A diagnostic message is not transmitted.

12. Show that the premises "One student in this class knows how to write programs in JAVA" and "everyone who knows how to write programs in JAVA can get high paying job." Imply the conclusion "someone in class can get a high paying job."