UNIT-III GRAPH THEORY

PART-B

DRAWING GRAPHS FROM GIVEN CONDITIONS

1. Draw the complete graph K5 with vertices A, B, C, D, E. Draw all complete subgraphs of K5 with 4 vertices.

2. Draw a graph with 5 vertices A, B, C, D, E such that deg(A) = 3, B is an odd vertex, deg(c) = 2, and D and E are adjacent.

3. Draw the graph with 5 vertices A, B, C, D, E such that deg(A) = 3, B is odd vertex, deg(c) = 2 and D and E are adjacent.

4. Determine which of the following graphs are bipartite and which are not. If a graph is bipartite, state if it is completely bipartite.

5. Find all connected subgraphs obtained from the graph given in the following figure, by deleting each vertex. List out the simple paths from A.

ISOMORPHISM OF GRAPHS:

1. Determine whether the graphs G and H given below are isomorphic.

2. Using circuits, examine whether the following pairs of graphs G1, G2 given below are isomorphic or not.
3. Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reason.

4. Check whether the two graphs given are isomorphic or not.

5. Determine whether the following graphs G and H are isomorphic.

6. The adjacency matrices of two pairs of graphs are given below. Examine the isomorphism of G and H by finding a permutation matrix.

\[ A_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A_H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

**General Problems in Graphs**

1. How many paths of length 4 are there from A to D in the simple graph G, given below?

2. Find an Euler path or an Euler circuit, if it exists, of the three graphs below. If it does not exist, explain why.
Which of the following simple graphs have a Hamiltonian circuit or, if not, a Hamiltonian path?

Check whether the graph given below is Hamiltonian or Eulerian or 2-colorable. Justify your answer.

Theorems
1. Prove that an undirected graph has an even number of vertices of odd degree.  
2. Prove that the maximum number of edges in a simple disconnected graph $G$ with $n$ vertices and $k$ components is $\frac{(n-k)(n-k+1)}{2}$.  
3. Show that if a graph with $n$ vertices is self-complementary, then $n \equiv 0 \text{ or } 1 \pmod 4$.  
4. Show that a graph $G$ is disconnected if and only if its vertex set $V$ can be partitioned into two nonempty subsets $V_1, V_2$ such that there exists no edge in $G$ whose one end vertex is in $V_1$ and the other in $V_2$.  
5. If all the vertices of an undirected graph are each of degree $K$, show that there exists no edge whose one is a multiple of $K$.  
6. Let $G$ be a simple undirected graph with adjacency matrix $A$ with respect to the ordering $V_1, V_2, V_3, \ldots, V_n$. Prove that the number of different walks of length $y$ from $V_i$ to $V_j$ equals the entry $A_y^{ij}$ of $A^y$, where $y$ is a positive integer.
THEOREMS BASED ON EULER AND HAMILTON GRAPH

1. Prove that a connected graph G has an Eulerian path if and only if all the vertices have even degree. [ML12] [NL13] [ML14]

2. Show that the complete graph with n vertices Kn has a Hamiltonian circuit whenever n > 3. [NL12]

3. Prove that if G is a simple graph with at least three vertices and \( \sum_{v \in V(G)} \deg(v) \geq 2|V(G)| \) then G is Hamiltonian. [ML13]

4. Let G be a simple undirected graph with n vertices, let u, v be two non-adjacent vertices in G such that \( \deg(u) + \deg(v) \geq n \) in G. Show that G is Hamiltonian if G + uv is Hamiltonian. [AML12]

THEOREMS AND THEORETICAL QUESTIONS

1. Let \( \delta(G) \) and \( \Delta(G) \) denote the minimum and maximum degrees of all the vertices of G, respectively. Then show that for a non-directed graph G, \( \delta(G) \leq \frac{1}{2}|E(G)| \leq \Delta(G) \). [AML15]

2. Let G be a graph with n vertices and m edges such that all the vertices have degree k or k+1. Prove that if G has nk vertices of degree k and nk+1 vertices of degree k+1, then \( n_k = \frac{(k+1)n - 3m}{2} \). [NL14]

3. Prove that a connected graph has an Euler Trail if and only if it has at most two vertices of odd degree. [NL15] [NL16]

4. Show that the complete graph Kn contains \( \frac{(n-1)!}{2} \) different Hamiltonian cycles. [NL12]

5. Prove that any circuit in a graph must contain a cycle and that any circuit which is not a cycle contains at least two cycles. [AML12]

6. Briefly explain about subgraph.

7. Briefly explain about walks with an example.
# UNIT-3 GRAPHS

- Graphs and Graphs Model
- Graph Terminology
- Special Types of Graphs
- Matrix Representation of Graphs
- Graph Isomorphism
- Connectivity
- Eulerian Graph
- Hamiltonian Graph

## Definitions

<table>
<thead>
<tr>
<th>No.</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><strong>Graph</strong>: A graph consists of two things</td>
<td><img src="image" alt="Graph Example" /></td>
</tr>
<tr>
<td></td>
<td>1. A set ( V = V(G) ) whose elements are called vertices, points or nodes.</td>
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<tr>
<td></td>
<td>2. A set ( E = E(G) ) of unordered pairs of distinct vertices called edges of ( G ), denoted by ( G(V,E) ).</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td><strong>Adjacent</strong>: Vertices ( U ) and ( V ) are said to be adjacent if there is an edge ( = U,V ).</td>
<td><img src="image" alt="Adjacent Example" /></td>
</tr>
<tr>
<td>3</td>
<td><strong>Incident</strong>: Edge ( e ) is said to be incident on each of its end points ( u ) and ( v ).</td>
<td><img src="image" alt="Incident Example" /></td>
</tr>
<tr>
<td>4</td>
<td><strong>Adjacent Edges</strong>: Two edges are said to be adjacent if they are incident on a common vertex.</td>
<td><img src="image" alt="Adjacent Edges Example" /></td>
</tr>
<tr>
<td>5</td>
<td><strong>Self Loop</strong>: If there is an edge from ( V ) to ( V ), then that edge is called a self loop.</td>
<td><img src="image" alt="Self Loop Example" /></td>
</tr>
<tr>
<td>6</td>
<td><strong>Parallel Edges</strong>: If two edges have same end points then the edges are parallel edge.</td>
<td><img src="image" alt="Parallel Edges Example" /></td>
</tr>
<tr>
<td>7</td>
<td><strong>Simple Graph</strong>: A graph which has neither self loops nor parallel edge is Simple graph.</td>
<td><img src="image" alt="Simple Graph Example" /></td>
</tr>
<tr>
<td>8</td>
<td><strong>Isolated Vertex</strong>: A vertex having no edge incident on it is an isolated vertex.</td>
<td><img src="image" alt="Isolated Vertex Example" /></td>
</tr>
<tr>
<td>9</td>
<td><strong>Multigraph</strong>: A graph which has more than one edge between a pair of vertices is Multigraph.</td>
<td><img src="image" alt="Multigraph Example" /></td>
</tr>
<tr>
<td>10</td>
<td><strong>Pseudograph</strong>: A graph in which loops and parallel edges are allowed is called Pseudograph.</td>
<td><img src="image" alt="Pseudograph Example" /></td>
</tr>
</tbody>
</table>
**WEIGHTED GRAPH**: A graph \( G \) in which weights are assigned to every edge is called a weighted graph.

**FINITE GRAPH**: A graph \( G = (V,E) \) in which both \( V \) \((V(n))\) and \( E \) \((E(n))\) are finite set is called a finite graph.

**DIRECTED GRAPH/DIGRAPH**: A graph in which every edge is directed is called a digraph.

**UNDIRECTED GRAPH**: A graph in which an edge is associated with an unordered pair of vertices is called an undirected graph.

**MIXED GRAPH**: If some edges are directed and some are undirected in a graph, the graph is called mixed graph.

**UNDERLYING UNDIRECTED GRAPH**: A graph obtained by ignoring the direction of edges in a directed graph is called the underlying graph.

**UNDERLYING SIMPLE GRAPH**: A graph obtained by deleting all loops and parallel edges from a graph is called a underlying simple graph.

<table>
<thead>
<tr>
<th>NO</th>
<th>TYPE</th>
<th>EDGES</th>
<th>MULTIPLE EDGES</th>
<th>LOOPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SIMPLE GRAPH</td>
<td>Undirected</td>
<td>NO</td>
<td>HD</td>
</tr>
<tr>
<td>2</td>
<td>MULTI GRAPH</td>
<td>Undirected</td>
<td>YES</td>
<td>HD</td>
</tr>
<tr>
<td>3</td>
<td>PSEUDOGRAPH</td>
<td>Undirected</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>4</td>
<td>SIMPLE DIRECTED GRAPH</td>
<td>Directed</td>
<td>NO</td>
<td>HD</td>
</tr>
<tr>
<td>5</td>
<td>DIRECTED MULTIGRAPH</td>
<td>Directed</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>6</td>
<td>MIXED GRAPH</td>
<td>Directed, Undirected</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>
**Exercise:**

1. Describe formally the below graph:

   **Sol:**
   - Graph G is an undirected graph.
   - Graph G is a pseudograph.
   - \( V(G) = \{ v_1, v_2, v_3, v_4, v_5, v_6 \} \)
   - \( E(G) = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7 \} \)
   - Adjacent vertices:
     \[ \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6)\} \]
   - Not adjacent vertices:
     \[ \{(v_1, v_3), (v_2, v_5), (v_4, v_6)\} \]
   - Loop = \( e_6 \).
   - Parallel edges = \( \{e_5, e_6\} \).
   - Adjacent edges:
     \[ \{(e_1, e_2), (e_2, e_3), (e_2, e_4), (e_4, e_5), (e_3, e_4)\} \]
   - Isolated vertex = \( \{v_6\} \).
   - Incident edges:
     \[ \{e_1 \text{ incident with } v_1, v_2; e_2 \text{ incident with } v_2, v_3; e_3 \text{ incident with } v_2, v_4; e_4 \text{ incident with } v_4, v_5; e_5 \text{ incident with } v_4, v_6\} \]
   - Not incident edges = \( \{e_6\} \).

2. What type of graph are the following?

   **Sol:**
   - (a) Simple graph
   - (b) Multiple graph
   - (c) Pseudograph
   - (d) Simple directed
   - (e) Directed multigraph
   - (f) Mixed graph

**Graph Terminology and Special Types of Graphs**

**Definition:** Degree of a vertex.

The number of edges incident at the vertex \( v_i \) is called the degree of the vertex with self-loops counted twice and it is denoted by \( d(v_i) \).

* A vertex of degree 0, \( 0 \leq d(v_i) = 0 \) is called **isolated**.
* A vertex of degree 1, \( 1 \leq d(v_i) = 1 \) is called **pendent**.
1. Find the degree of the vertices of the undirected graph.

(a) \( \begin{align*} 
& d(V_1) = 3, \\
& d(V_2) = 1, \\
& d(V_3) = 2, \\
& d(V_4) = 1 
\end{align*} \)

(b) \( \begin{align*} 
& d(a) = 2, \\
& d(b) = 1, \\
& d(c) = 1, \\
& d(d) = 1 
\end{align*} \)

IN DEGREE AND OUT-DEGREE OF DIRECTED GRAPH:

IN-DEGREE OF \( V \) (\( deg^-(V) \)):
The no. of edges with \( V \) as their terminal vertex.

OUT-DEGREE OF \( V \) (\( deg^+(V) \)):
The no. of edges with \( V \) as their initial vertex.

NOTE:
A loop at a vertex contributes 1 to both indegree & outdegree.

2. Find the degree of the directed graph.

(a) \( \begin{align*} 
& d^-(a) = 3, \\
& d^-(b) = 1, \\
& d^-(c) = 2, \\
& d^-(d) = 1 
\end{align*} \)

(b) \( \begin{align*} 
& d^-(V_1) = 1, \\
& d^-(V_2) = 1, \\
& d^-(V_3) = 1, \\
& d^-(V_4) = 1 
\end{align*} \)

<table>
<thead>
<tr>
<th>IN DEGREE</th>
<th>OUT DEGREE</th>
<th>TOTAL DEGREE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d^-(a) = 3 )</td>
<td>( d^+(a) = 1 )</td>
<td>( d(a) = 4 )</td>
</tr>
<tr>
<td>( d^-(b) = 1 )</td>
<td>( d^+(b) = 2 )</td>
<td>( d(b) = 3 )</td>
</tr>
<tr>
<td>( d^-(c) = 2 )</td>
<td>( d^+(c) = 1 )</td>
<td>( d(c) = 3 )</td>
</tr>
<tr>
<td>( d^-(d) = 1 )</td>
<td>( d^+(d) = 3 )</td>
<td>( d(d) = 4 )</td>
</tr>
</tbody>
</table>

THEOREM 1: THE HANDSHAKING THEOREM (\( \times 8M \))

Let \( G = (V, E) \) be an undirected graph with edges. Then \( \sum_{v \in V} deg(v) = 2 \times \) where \( E \) is number of edges.

The sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph, hence even.

**Proof:** Let \( G = (V, E) \) be any graph, where \( V = \{ V_1, V_2, \ldots, V_n \} \) & \( E = \{ e_1, e_2, \ldots \} \).

Since each edge contributes exactly 2 to sum of the degree spectrum.
THEOREM 2: \( \text{[Should Prove Thm 1 also]} \)

If a directed graph, then the number of odd degree vertices are even.

**Proof:**

Let \( V_1 \) be the set of vertices of even degree.

Let \( V_2 \) be the set of vertices of odd degree.

By Handshaking Theorem,

\[
\sum_{v \in V} \deg(v) = 2e
\]

For vertices in \( V_1 \),

\[
\sum_{v \in V_1} \deg(v) = \text{even}
\]

For vertices in \( V_2 \),

\[
\sum_{v \in V_2} \deg(v) = \text{even}
\]

Therefore, the number of odd degree vertices are even.

THEOREM 3:

If \( G = (V, E) \) be a directed graph, then \( \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |E| \)

**Proof:**

Consider an edge \( e = (u, v) \) it is incident into \( v \) and one into \( u \).

Similarly, every edge contributes to the sum of indegrees and outdegrees of the vertices.

The sum of indegrees = sum of outdegrees = \( |E| \).

The maximum number of edges in a simple graph with \( n \) vertices is \( \frac{n(n-1)}{2} \).

**Proof:**

By Handshaking theorem,

\[
\sum_{v \in V} \deg(v) = 2e
\]

The number of edges with \( n \) vertices in graph \( G \).

\[
\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2e
\]

\[
\sum_{i=1}^{n} \deg(v_i) = 2e \quad \text{[The max deg of each vertex is \( n-1 \)]}
\]

\[
(n-1) + (n-1) + \cdots + (n-1) = 2e \quad \text{[\( n \) times]}
\]

\[
\Rightarrow n(n-1) = 2e \quad \Rightarrow e = \frac{n(n-1)}{2}
\]

Hence, the max no. of edges in any simple graph is \( \frac{n(n-1)}{2} \).
Problems.

1. Let \( S(n) \) and \( \Delta(n) \) denote minimum and maximum degrees of all the vertices of \( G \) respectively. Then show that for a non-directed graph \( G \), \( S(n) \leq 2|E| \leq \Delta(n) \).

So. \( S(n) = \min \) degree of all vertices of \( G \).
\( \Delta(n) = \max \) degree of all vertices of \( G \).
\( |V| = \) no. of vertices of graph \( G \).
\( |E| = \) no. of edges of graph \( G \).

By Handshaking theorem, \( \sum_{V} \deg (V) = 2|E| \Rightarrow 0 \)

\( \min \left( \sum_{V} \deg (V) \right) = S(n) + S(n) + \cdots + |V| \text{ times} = |V| S(n) \Rightarrow 2 \)

\( \max \left( \sum_{V} \deg (V) \right) = \Delta(n) + \Delta(n) + \cdots + |V| \text{ times} = |V| \Delta(n) \Rightarrow 3 \)

From (i) and (ii) \( |V| S(n) \leq 2|E| \Rightarrow \frac{|E|}{|V|} \leq S(n) \Rightarrow 4 \)

From (ii) and (iii) \( 2|E| = |V| \Delta(n) \Rightarrow \frac{|E|}{|V|} \leq \Delta(n) \Rightarrow 5 \)

From (i) and (iv) \( S(n) \leq \frac{|E|}{|V|} \leq \Delta(n) \). Hence it is proved.

2. Let \( G \) be a graph with \( n \) vertices and \( n \) edges, such that the vertices have degree \( k \) or \( k+1 \). Prove that if \( G \) has \( k \) vertices of degree \( k \) and \( N_{k+1} \) vertices of degree \( k+1 \), then \( N_k = C(k+1) + 2n \).

Sol. Given a graph \( G \) in which \( |V| = n \) and \( |E| = n \).

The no. of vertices of degree \( k = N_k \Rightarrow 0 \)

The no. of vertices of degree \( k+1 = N_{k+1} \Rightarrow 0 \)

So, \( N_k + N_{k+1} = n \Rightarrow N_{k+1} = n - N_k \).

By fundamental theorem of graph theory, \( \sum \deg (V) = 2|E| \Rightarrow 0 \)

From (i) and (ii) \( k \cdot N_k + (k+1) (n - N_k) = 2n \).

Hence it is proved.

3. If all the vertices of an undirected graph are each of degree \( k \), show that the number of edges of the graph is a multiple of \( k \).

Sol. Let \( 2n \) be the number of vertices of the given graph.

Let \( |E| = \) the no. of edges of the given graph.

By Handshaking theorem, \( \sum \deg (V) = 2|E| \Rightarrow 0 \)

\( \Rightarrow |E| = \frac{k \cdot N_k}{2} = \frac{N_k \cdot n}{2} \Rightarrow 1E = n \cdot k \).

Hence, the number of edges of the given graph is a multiple of \( k \).
5. How many edges are there in a graph with 10 vertices each of degree 6?
Solution: Let e be the number of edges of the graph.
\[ 2e = \text{Sum of all degree} = 10 \times 6 = 60 \]
\[ e = 30 \]
There are 30 edges in a graph with 10 vertices each of degree 6.

6. Can a simple graph exist with 15 vertices of degree 5?
Solution: Let k - T be deg \((v) = 2k \Rightarrow 2e = 15 \times 5 = 75 \neq \text{Even} \]
Such a graph is not possible.

7. Does there exist a simple graph with 5 vertices of the following degree? If so, draw such a graph.
Solution:
(a) 1, 1, 1, 1, 1
Degree sequence: (1, 1, 1, 1, 1)
No. - A star is always odd degree vertices is always even.
Here, there are 5 odd degree vertices.
Such a graph does not exist.
(b) 3, 3, 3, 3, 2
Degree sequence: (3, 3, 3, 3, 2),
Here, there are 4 odd degree vertices.
Such a graph exists.

8. Determine n for the following graphs.
(a) G has 9 edges and all vertices have degree 3.
Solution: Let \( |V| = x \) and each vertex has degree 3.
\[ \sum_{v \in V} \deg(v) = 2e \]
\[ 3x = 2(9) \Rightarrow x = 6 \]
There are 6 vertices.

(b) G is regular with 15 edges.
Solution: Let \( |V| = x \), Since G is regular, all vertices have the same degree r.
By Handshaking theorem, \( \sum_{v \in V} \deg(v) = 2e \)
\[ 2x = 15 \times 15 = 30 \]
x = 15\( \Rightarrow x = 15 \times \frac{15}{15} = 15 \times 1 \)
(c) G is regular of degree 4 with 10 edges.
Solution: Let \( |V| = n \) then
\[ \leq \deg(v) = 4n \]
\[ 2 \times 10 = 4n \]
\[ n = 5 \]
5. Find the number of vertices, no. of edges and the degrees of each vertex and also verify the handshaking theorem.

\[ \begin{align*}
&v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \\
&d(v_1) = 4, d(v_2) = 5, d(v_3) = 6, d(v_4) = 3 \\
&d(v_5) = 4, d(v_6) = 3, d(v_7) = 5, d(v_8) = 3 \\
&\text{deg}(v) = 2\text{deg}(E) \Rightarrow 4 + 5 + 6 + 3 + 4 + 5 + 3 = 2 \times 15 \\
&\Rightarrow 30 = 30 \\
\end{align*} \]

- Handshaking theorem is verified.

### SPECIAL TYPES OF GRAPHS

1. **Complete Graph, \( K_n \):**
   - No. of vertices: \( n \)
   - No. of edges: \( \frac{n(n-1)}{2} \)
   - Degree sequence: \( (n-1, n-1, \ldots, n-1) \).

A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge.

2. **Cycle, \( C_n \):**
   - No. of vertices: \( n \)
   - No. of edges: \( n \)
   - Degree sequence: \( (2, 2, 2, \ldots) \)

The cycle \( C_n \), \( n \geq 3 \) consists of \( n \) vertices \( 1, 2, \ldots, n \) and \( n \) edges \( \{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\} \).

3. **Wheel, \( W_n \):**
   - No. of vertices: \( n + 1 \)
   - No. of edges: \( 2n \)
   - Degree sequence: \( (n, 3, 3, 3, \ldots) \)

The wheel, \( W_n \), is obtained by adding an additional vertex to the cycle \( C_n \), for \( n \geq 3 \) and connecting this new vertex to each of the \( n \) vertices in \( C_n \) by new edges.

4. **n-Cubes, \( Q_n \):**
   - No. of vertices: \( 2^n \)
   - No. of edges: \( n \cdot 2^{n-1} \)
   - Degree sequence: \( (n, n, n, \ldots) \)

The \( n \)-dimensional hypercube, \( Q_n \), is the graph that has vertices representing \( 2^n \) bit strings of length \( n \).
A complete bipartite graph is a bipartite graph in which every vertex of \( V_i \) is adjacent to every vertex of \( V_j \).

**Regular Graph, \( K \):**
If every vertex of a simple graph has the same degree, then the graph is called a regular graph. (OP) No. of vertices and no. of edges are same.

### 2-Regular Graphs

### 3-Regular Graphs

The union of a simple graph \( G_1 = [V_1, E_1] \) & \( G_2 = [V_2, E_2] \) is the simple graph with vertex set \( V_1 \cup V_2 \) & edge set \( E_1 \cup E_2 \).

**Subgraph:**
Given two graphs \( G_1 \) & \( G_r \), we say that \( G_1 \) is a subgraph of \( G_r \) if the following conditions hold:

- All the vertices and all the edges of \( G_1 \) are in \( G_r \).
- Each edge of \( G_1 \) has the same end vertices in \( G_r \) as in \( G_1 \).

**Example:** \( G_1 \) is a subgraph of \( G_r \) since all vertices \& all edges of the graph \( G_1 \) are in the graph \( G_r \) & that every edge in \( G_1 \) has the same end vertices in \( G_r \) as in \( G_1 \).

**Results on Subgraph:**
1. Every graph is a subgraph of itself.
2. Every simple graph of \( n \) vertices is a subgraph of Complete graph \( K_n \).
3. If \( G_1 \) is a subgraph of \( G_2 \), \( G_2 \) is a subgraph of \( G_1 \), then \( G_1 \) is a subgraph of \( G_1 \).
4. A single vertex in a graph \( G \) is a subgraph of \( G \).
5. Any single edge in a graph \( G \), together with its end vertices, is a subgraph of \( G \).
**COMPLEMENTARY GRAPH:**

The complement \( \overline{G} \) of a simple graph \( G \) has the same vertices and edges. Two vertices are adjacent in \( \overline{G} \) iff they are not adjacent in \( G \).

**Example:** If \( G = K_n \) then \( \overline{G} \) is a graph with \( n \) vertices and no edges.

Notation:
- \( G \) is a self-complementary graph if \( \overline{G} = G \).
- A graph \( G \) is said to be self-complementary if \( \overline{G} \equiv G \).

**Theorem 5:**

Show that if \( G \) is self-complementary then \( n \equiv 0,1 \) (mod 4).

Any self-complementary graph has \( 4n \) or \( 4(n+1) \) vertices, respectively.

A simple graph \( G \) on \( n \) vertices is self-complementary only if either \( n \) or \( n-1 \) is divisible by 4.

**Proof:**

\( G_m \), \( G \) is self-complementary \( \Rightarrow \) No. of edges in \( G \) = No. of edges in \( \overline{G} \)

\[ |E(G)| = |E(\overline{G})| = 0 \]

Also by K.T., total no. of edges possible = No. of edges in \( G \) + No. of edges in \( \overline{G} \) with \( n \) vertices

\[ n(n-1) \]

\[ = m + m = 2m \]

\[ = n(n+1) = 4m \Rightarrow n(n-1) \text{ is multiple of 4.} \]

Either \( n \) or \( n-1 \) is divisible by 4.

\( \therefore \) \( G \) is self-complementary simple graph with \( n \) vertices then \( n \equiv 0,1 \) (mod 4).

**Theorem 6:**

A graph is bipartite if and only if it contains no odd cycle.

**Proof:**

Suppose \( G \) is a bipartite graph. Let \( X \) and \( Y \) be the partition of vertices \( S \).

If \( G \) has no odd cycle:

Suppose \( C = V_0, V_1, V_2, \ldots, V_k, V_0 \) be a cycle of \( G \).

**Assume**

- \( V_0V_2 \in E \) \( \Rightarrow V_1 \notin V \)
- \( V_1V_2 \in E \) \( \Rightarrow V_0 \notin V \)
- \( V_2V_3 \in E \) \( \Rightarrow V_1 \notin V \)
- \( V_3V_1 \in E \) \( \Rightarrow V_2 \notin V \)
- \( V_4V_3 \in E \) \( \Rightarrow V_0 \notin V \)
- \( \vdots \)
- \( V_{k}V_{k-1} \in E \) \( \Rightarrow V_{k-2} \notin V \)
- \( V_{k-1}V_{k} \in E \) \( \Rightarrow V_{k-2} \notin V \)

Let we see that vertices of the cycle with odd suffixes are in \( X \) and even suffixes are in \( Y \).
\[ v_0 \in X \land v_k \in \bar{X} \Rightarrow \forall x,y \\
\Rightarrow k \text{ is odd} \\
\Rightarrow \bar{k} \text{ is even.} \]

Note that length of \( C \) is \( (k+1) \). Every cycle is an even cycle.

Conversely,

Suppose \( G \) be a graph which contains no odd cycle.

\( \overline{\text{If } G \text{ is Bipartite}}. \)

Assume that \( G \) is a connected graph having no odd cycle.

We choose an arbitrary vertex \( u \in G \) and define

\[ X = \{ x \in V \mid d(u,x) \text{ is even}\} \]

\[ Y = \{ y \in V \mid d(u,y) \text{ is odd}\}. \]

Clearly \( (X,Y) \) is a partition of \( V \).

We shall show that \( U(C^X,Y) \) is a bipartition of \( G \).

Suppose that \( v,w \) are two vertices of \( X \).

Let \( P \) be a shortest \((u,v)\) path and

\( Q \) be a shortest \((u,w)\) path.

\( u \) be the last vertex common to both \( P \) and \( Q \).

Since \( P \) and \( Q \) are shortest path \( S \).

\( (u,u) \)-sections of both \( P \) and \( Q \) are shortest \((u,u)\) path.

Have the same length.

\( \forall v \in X \Rightarrow \) The length of \( P \) is even.

\( \forall v \in X \Rightarrow \) The length of \( Q \) is even.

\( \Rightarrow \) The length of \((u,v)\) \& length of \((u,w)\) must have same length.

\( \Rightarrow (v,u) \) path \( P_1 \cup Q \) is of even length.

If there is an edge \( v_0 \in E(G) \) then \( P_1 \cup Q \) is a cycle of odd length.

Which is contradiction to our assumption.

\( \therefore \) \( v_0 \in \bar{E} \).

Hence, no two vertices in \( X \) are adjacent.

\( \forall y \in Y \) there is no two vertices in \( Y \) are adjacent.

Every edge of \( G \) joins a vertex of \( X \) to a vertex of \( Y \).

\( (X,Y) \) is a partition of \( G \).

Hence \( G \) is Bipartite.
1. Show that $C_6$ is bipartite?

Sol:

The vertex set of $C_6$ can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of $C_6$ connects a vertex in $V_1$ and a vertex in $V_2$. Hence, $C_6$ is a Bipartite Graph.

2. Is $K_3$ a Bipartite?

Sol: No, the Complete graph $K_3$ is not Bipartite.

If we divide the vertex set of $K_3$ into two disjoint sets, one of the two sets must contain two vertices.

If graph is Bipartite, these two vertices should not be connected by an edge, but in $K_3$ each vertex is connected to every other vertex by an edge.

3. Are the following graphs $G$ & $H$ Bipartite?

Sol:

Graph $G$ is Bipartite, because its vertex set is the union of two disjoint sets $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$ and each edge connects a vertex in one of these subsets to a vertex in the other subset.

Graph $H$ is not Bipartite, because its vertex set cannot be partitioned into two subsets so that edges do not connect 2 vertices from the same subset.

4. Show the different Subgraphs of the graph

Sol: Different Subgraphs are:

- $G_1$
- $G_2$
- $G_3$
- $G_4$
- $G_5$
- $G_6$
- $G_7$

Let $G$ be a bipartite graph with edges $e_1, e_2$.

Let $V_1 = m$, $V_2 = n$.

W.K.T, the no. of edges of a bipartite graph is max. when it is complete bipartite graph $K_{m,n}$ so the maximum no. of edges is $mn$.

The no. of edges of $G$ is $e = e_1 + e_2$.

We know, $\frac{m+n}{2} \leq \sqrt{mn} \quad \text{[\text{-} AM \geq GM]}$

$\Rightarrow \frac{(m+n)^2}{4} \geq mn \Rightarrow e \leq \frac{m+n}{2}$ \quad \text{[using (1)]}

$\Rightarrow \frac{\sqrt{e}}{2} \geq e \Rightarrow e \leq \frac{\sqrt{e}^2}{4}$

**Representing Graphs & Graph Isomorphism**

1. **Adjacency Matrix**

   **Adjacency Matrix for Simple Undirected Graph**

   The adjacency matrix for the undirected graph $G$ is denoted by $A = [a_{ij}]_{n \times n}$.

   $a_{ij} = \begin{cases} 1, & \text{if there is an edge between } v_i \text{ and } v_j \\ 0, & \text{otherwise} \end{cases}$

   All undirected graphs including multigraphs and pseudographs have symmetric adjacency matrices, i.e., $a_{ij} = a_{ji}$.

   **Observations:**
   - The sum of all the entries in any row is equal to the degree of the vertex corresponding to that row.
   - All the entries along the leading diagonal are zero, and only if the graph has no self-loop.
   - The $(i, j)^{th}$ entry of $A^n$ is the number of different paths of length $n$ between vertices $v_i$ and $v_j$.
   - Given any square, symmetric, binary (0,1) matrix $A$ of order $n$, each of the $n$ adjacency matrices

2. **Incidence Matrix**

3. **Path Matrix**
Problems:

1. Find the adjacency matrices of the following undirected graphs.

(a) 
(b) 
(c) 

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \]

2. Find the adjacency matrices of the following directed graphs.

(a) 
(b) 
(c) 

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \]

3. Draw directed graphs to the following adjacency matrices.

(a) 
(b) 

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \]
Graphs to the following adjacency matrices:

\[
A = \begin{bmatrix}
1 & 3 & 2 \\
3 & 0 & 4 \\
2 & 4 & 0 \\
\end{bmatrix}
\]

\[
A_e = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

2. **INCIDENCE MATRIX**

Let \( G = (V, E) \) be an undirected graph with \( n \) vertices \( \{v_1, v_2, \ldots, v_n\} \) and \( m \) edges \( \{e_1, e_2, \ldots, e_m\} \). Then the \((n \times m)\) matrix \( B = [b_{ij}] \) where

\[
b_{ij} = \begin{cases} 
1, & \text{when edge } e_j \text{ incident on vertex } v_i \\
0, & \text{otherwise} 
\end{cases}
\]

**Observations:**

1. Since each edge is incident on exactly two vertices, each column in \( B \) has exactly two 1's.
2. The sum of the entries in any one row gives the degree of the vertex corresponding to that row.

**Problems**

1. Find the incidence matrices for the following graphs:

   (a) \[
   A = \begin{bmatrix}
   v_1 & e_1 & e_2 & e_3 & e_4 & e_5 \\
   v_2 & 1 & 1 & 0 & 0 & 1 \\
   v_3 & 1 & 0 & 0 & 0 & 0 \\
   v_4 & 0 & 1 & 1 & 0 & 1 \\
   \end{bmatrix}
   \]

2. (b) \[
   B = \begin{bmatrix}
   v_1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
   v_2 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
   v_3 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
   v_4 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
   v_5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
   \end{bmatrix}
   \]

3. Find the incidence matrices for the graphs

   (i) \( K_4 \)  (ii) \( W_5 \)  (iii) \( C_6 \)  (iv) \( K_3 \)

4. Find the adjacency matrices for the following graphs

   (a) \( K_5 \)  (ii) \( K_{3,1,5} \) (iii) \( C_5 \) (iv) \( W_4 \)  (v) \( Q_3 \)
2. **Path Matrix or Reachability Matrix:**

Let \( G = (V, E) \) be a simple digraph in which \( |V| = n \) & the nodes of \( G \) are assumed to be ordered. An \( n \times n \) matrix \( P \) whose elements are given by

\[
P_{ij} = \begin{cases} 
1 & \text{if there exists a path from } V_i \text{ to } V_j \\
0 & \text{otherwise}
\end{cases}
\]

**Problems**

1. Find path matrix of

   **Solution:**

   **Path Matrix:**

   \[
   B = [P_{ij}] = \begin{bmatrix}
   1 & 1 & 1 & 0 \\
   1 & 1 & 1 & 1 \\
   0 & 0 & 0 & 0 \\
   1 & 1 & 1 & 1
   \end{bmatrix}
   \]

2. Find path matrix \( P(\{v_2, v_4\}) \) for the following graph \( G \). What is the observation on matrix \( P(v_2, v_4) \)?

   **Solution:** There are 3 different paths from \( v_2 \) to \( v_4 \).

   **Paths:** \( \{e_1, e_3\} \), \( \{e_1, e_2, e_3\} \), \( \{e_2, e_3\} \)

   \[
P(v_2, v_4) = P_1 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 \\
   1 & 0 & 0 & 0 & 1 & 0 \\
   0 & 1 & 1 & 0 & 0 & 0
   \end{bmatrix}
   \]

   **Observations**

   (i) A column of all 0s will correspond to an edge that does not lie on any path between \( V_i \) to \( V_j \).

   (ii) A column of all 1s will correspond to an edge that lies in every path between \( V_i \) to \( V_j \).

3. Find the adjacency matrix of the following graph \( G \). Hence find degree of each vertex. Also find \( A^2 \) & \( A^3 \). What is your observation regarding the entries in \( A^2 \) & \( A^3 \)?

   **Solution:**

   **Adjacency Matrix:**

   \[
   A = \begin{bmatrix}
   0 & 1 & 0 & 1 \\
   1 & 0 & 1 & 0 \\
   0 & 1 & 0 & 0 \\
   1 & 0 & 1 & 0
   \end{bmatrix}
   \]

   \[
   \begin{align*}
   \text{deg}(v_1) &= \text{sum of the entries of 1st row} = 2 \\
   \text{deg}(v_2) &= \text{sum of the entries of 2nd row} = 3 \\
   \text{deg}(v_3) &= \text{sum of the entries of 3rd row} = 2 \\
   \text{deg}(v_4) &= \text{sum of the entries of 4th row} = 3
   \end{align*}
   \]
\[
A^2 = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]
\[
A^3 = A^2 \cdot A = \begin{bmatrix}
2 & 1 & 2 & 1 \\
1 & 3 & 1 & 2 \\
2 & 1 & 2 & 1 \\
1 & 2 & 1 & 3
\end{bmatrix},
\]
\[
A^4 = A^3 \cdot A = \begin{bmatrix}
2 & 5 & 2 & 5 \\
5 & 4 & 5 & 5 \\
2 & 5 & 2 & 5 \\
5 & 5 & 5 & 4
\end{bmatrix}.
\]

**Observations:**
1. \( A^2 \) and \( A^3 \) are symmetric matrices.
2. \((1,1)^{th}\) entry of \( A^2 \) = degree of \( v_1 \),
   \((2,2)^{th}\) entry of \( A^2 \) = degree of \( v_2 \),
   \((3,3)^{th}\) entry of \( A^2 \) = \( d(v_3) \)
3. \((1,1)^{th}\) entry of \( A^2 \) = \{ number of different paths of lengths 2 between \( i^{th} \) and \( j^{th} \) vertices.
4. \((1,3)^{th}\) entry of \( A^2 \) = \{ no. of different paths of length 2 between \( v_1 \) & \( v_3 \).

Find the adjacency matrix of the following graph. Find \( A^2, A^3 \) and \( Y = A + A^2 + A^3 + A^4 \). What is your observation about entries in \( A^2 \) & \( A^3 \).

\[ Y = A + A^2 + A^3 + A^4 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2
\end{bmatrix}.

**Observations:**
1. \( A^2, A^3 \) are symmetric matrices.
2. \((1,1)^{th}\) entry of \( A^2 \) = degree of \( v_1 \),
   \((3,3)^{th}\) entry of \( A^2 \) = \( d(v_3) \)
3. \((1,1)^{th}\) entry of \( A^2 \) = \{ no. of different paths of length 2 between \( v_1 \) & \( v_3 \).
4. \((4,1)^{th}\) entry of \( A^2 \) = 0 \{ there is no path of length 2 between \( v_4 \) & \( v_1 \).
Consider the following digraph. Find the no. of possible elementary paths of length 3 from vertex $V_1$ to $V_2$.

Set the adjacency matrix of the graph $G$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Here $C_{1,2}^{3}$ entry of $A^3 = 2$.

There are 2 elementary paths of length 3 from vertex $V_1$ to $V_2$ namely

1. $V_1 \rightarrow V_3 \rightarrow V_4 \rightarrow V_2$
2. $V_1 \rightarrow V_2 \rightarrow V_1 \rightarrow V_2$

For the graph given below, find all possible paths of length 4 from vertex $B$ to $D$.

Set: The adjacency matrix, $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Now,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 3 & 4 & 1 & 3 \\ 4 & 3 & 2 & 2 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 3 & 4 & 1 & 3 \\ 4 & 3 & 2 & 2 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 4 & 4 \\ 6 & 6 & 4 & 4 \\ 6 & 6 & 4 & 4 \end{bmatrix}$$

The entry at $(2,4)$ in $A^4$ is 4. Hence there are 4 paths of length 4 from $B$ to $D$ namely:

1. $B \rightarrow A \rightarrow B \rightarrow A \rightarrow D$
2. $B \rightarrow A \rightarrow D \rightarrow C \rightarrow D$
3. $B \rightarrow A \rightarrow D \rightarrow A \rightarrow D$
4. $B \rightarrow A \rightarrow C \rightarrow A \rightarrow D$
Find the number of paths of length 4 from the vertex D to the vertex E in the following undirected graph. A: The adjacency matrix of the graph.

\[
A = \begin{bmatrix}
A & B & C & D & E \\
B & 0 & 1 & 0 & 1 \\
C & 1 & 0 & 1 & 0 \\
D & 0 & 1 & 0 & 0 \\
E & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Now,

\[
A^2 = A \cdot A = \begin{bmatrix}
2 & 0 & 2 & 0 & 1 \\
0 & 3 & 1 & 2 & 1 \\
2 & 1 & 3 & 0 & 1 \\
0 & 2 & 0 & 2 & 1 \\
1 & 1 & 1 & 1 & 2 \\
\end{bmatrix}
\]

Hence, the matrix \(A^4\), the \((4,5)\)th entry = 6. From the graph, these 6 paths are:

1. D → A → D → C → E
2. D → C → D → C → E
3. D → A → B → C → E
4. D → C → E → C → E
5. D → C → E → B → E
6. D → C → B → C → E

**GRAPH ISOMORPHISM**

Two graphs \(G\) and \(G'\) are isomorphic if there is a function \(f: V(G) \rightarrow V(G')\) from the vertices of \(G\) such that:

1. \(f\) is one-one
2. \(f\) is onto
3. \(f\) preserves adjacency

**If for vertices \(u, v\) in \(G\), \((u, v) \in E(G)\) if and only if \((f(u), f(v)) \in E(G')\), two graphs are isomorphic.**

If the two graphs have the same number of vertices and edges, they are isomorphic. Equal no. of vertices with same degree.

Check the given 2 graphs \(G\) and \(G'\) are isomorphic or not.

**Solution:**

\(G\): \(V = 5\), \(E = 6\)
\(G'\): \(V' = 5\), \(E' = 6\)

<table>
<thead>
<tr>
<th>Vertex of (G)</th>
<th>Degree</th>
<th>Vertex of (G')</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1)</td>
<td>2</td>
<td>(u_1)</td>
<td>2</td>
</tr>
<tr>
<td>(v_2)</td>
<td>2</td>
<td>(u_2)</td>
<td>2</td>
</tr>
<tr>
<td>(v_3)</td>
<td>2</td>
<td>(u_3)</td>
<td>2</td>
</tr>
<tr>
<td>(v_4)</td>
<td>2</td>
<td>(u_4)</td>
<td>2</td>
</tr>
<tr>
<td>(v_5)</td>
<td>2</td>
<td>(u_5)</td>
<td>2</td>
</tr>
</tbody>
</table>

Mapping:

\(v_1 \rightarrow u_1\), \(v_2 \rightarrow u_2\), \(v_3 \rightarrow u_3\), \(v_4 \rightarrow u_4\), \(v_5 \rightarrow u_5\)
Both G and G' have same vertices & same edge & same degree sequence. \( \therefore G \) is isomorphic to \( G' \).

\( \Box \) Check the given 2 graphs \( G \) & \( G' \) are isomorphic or not.

\( G : |V| = 5, |E| = 6 \) & \( G' : |V| = 5, |E| = 6 \)

<table>
<thead>
<tr>
<th>Vertex of ( G )</th>
<th>Degree</th>
<th>Vertex of ( G' )</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>2</td>
<td>( v_1 )</td>
<td>4</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>2</td>
<td>( v_2 )</td>
<td>2</td>
</tr>
<tr>
<td>( v_3 )</td>
<td>3</td>
<td>( v_3 )</td>
<td>3</td>
</tr>
<tr>
<td>( v_4 )</td>
<td>2</td>
<td>( v_4 )</td>
<td>2</td>
</tr>
<tr>
<td>( v_5 )</td>
<td>3</td>
<td>( v_5 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Here \( G \) & \( G' \) have same no. of vertices & edges. But degree sequence is not same. \( \therefore G \neq G' \) are not isomorphic.

\( \Box \) Are the graphs given in the following graph figure isomorphic?

\( G_1 : |V| = 8, |E| = 10 \)
\( G' : |V| = 8, |E| = 10 \)

<table>
<thead>
<tr>
<th>Vertex of ( G )</th>
<th>Degree</th>
<th>Vertex of ( G' )</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>2</td>
<td>( b_1 )</td>
<td>3</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>3</td>
<td>( b_2 )</td>
<td>2</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>2</td>
<td>( b_3 )</td>
<td>2</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>3</td>
<td>( b_4 )</td>
<td>3</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>2</td>
<td>( b_5 )</td>
<td>2</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>3</td>
<td>( b_6 )</td>
<td>3</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>2</td>
<td>( b_7 )</td>
<td>2</td>
</tr>
<tr>
<td>( a_8 )</td>
<td>3</td>
<td>( b_8 )</td>
<td>3</td>
</tr>
</tbody>
</table>

Here \( G \) & \( G' \) have same no. of vertices, edge & same degree sequences. But in \( G' \), the vertices \( b_1, b_4, b_5, b_8 \) are adjacent to more than one vertex of degree 3.
\( \therefore G \) & \( G' \) are not isomorphic.

\( \Box \) Determine whether the graphs given below are isomorphic or not?

\( G : |V| = 5, |E| = 9 \)
\( G' : |V| = 5, |E| = 9 \)

In \( G \), the vertices \( a, e, g \) each of degree 2 is not adjacent to any one of the vertex of degree 2.

But in \( G' \), the vertices \( v_2, v_3, v_6, v_7 \) are each of degree 2 adjacent to exactly one vertex of degree 2.
\( \therefore G \neq G' \) are not isomorphic.
Check the given two graphs $G$ & $G'$ are isomorphic or not.

$G$: Have both $G$ & $G'$ have the same vertex set and same number of edge with same degree sequence.

In $G$, vertices $B$, $C$ are of degree 2 and are adjacent to each other.

If $G$ & $G'$ were isomorphic, then the image of this vertex in $G'$ should be adjacent to both vertices of degree 2.

$G'$: In $G_1$ & $G_2$.

No. of vertices = 5 & No. of edges = 8

$G_1$: $d(u_1) = 3$
$d(u_2) = 4$
$d(u_3) = 2$
$d(u_4) = 4$
$d(u_5) = 2$

$G_2$: $d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 2$
$d(v_4) = 3$
$d(v_5) = 4$

By this choice of correspondence of vertices, we obtain the same adjacency matrix.

$A_1 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}$

By this correspondence of vertices, we obtain the same adjacency matrix.

$A_2 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$

$G_1$ & $G_2$ are isomorphic.

**Connectivity**

A path in a graph is a sequence $v_1, v_2, ..., v_k$ of vertices each adjacent to the next. (OR) Starting with the vertex $v_1$ one can travel along edges $(v_1, v_2)$, $(v_2, v_3)$ & reach the vertex $v_k$.

**Length of the Path**: The number of edges appearing in the sequence of a path is called the length of path.

**Cycle of Circuit**: A path which originates & ends in the same node is called a cycle of circuit.

**Simple Path**: A path is said to be simple if all the edges in the path are distinct.

**Elementary Path**: A path in which all the vertices are traversed only once is called an elementary path.
**Example:**

1. \( a \rightarrow d \rightarrow c \rightarrow f \rightarrow e \) → **Simple Path**
   - Length 4
2. \( a \rightarrow b \rightarrow e \rightarrow d \rightarrow a \) → **Not a Simple Path**
   - Length 5
3. \( b \rightarrow c \rightarrow f \rightarrow e \rightarrow b \) → **Cycle or Circuit**
   - Length 4
4. \( e \rightarrow b \rightarrow c \rightarrow f \rightarrow a \) → **Not a Path**
5. \( a \rightarrow b \rightarrow e \rightarrow a \rightarrow d \) → **Simple but Not Elementary Path**

**Connected Graph:** A graph is called connected if there is a path from any vertex to any other vertex or vice versa (or there is at least one path between every pair of vertices in the graph).

**Disconnected Graph:** A graph is called disconnected if there is no path between any two of its vertices.

**Connected Components of a Graph:**

The connected subgraphs of a graph \( G \) are called components of the graph \( G \).

A graph \( G \) that is not connected has two or more connected components that are disjoint and have \( G \) as their union.

**Cut Vertex:** The removal of a vertex from a connected graph produces a subgraph that is not connected. Such a variable is called a cut vertex.

**Cut Edge:** The removal of an edge from a connected graph produces a subgraph that is not connected.

**Example:** Find the cut edges & cut vertices in the graph.

- Cut vertex: \( b, c \) and \( e \)
- Cut edge: \( \{c, e\} \)
1. **Unilaterally Connected:** A simple digraph is said to be unilaterally connected if for any pair of nodes of the graph at least one of the nodes of the pair is reachable from the other node.

2. **Strongly Connected:** A directed graph $G$ is said to be strongly connected if there is a path from $u$ to $v$ and from $v$ to $u$ for any pair of vertices $u$ and $v$ in $G$.

3. **Weakly Connected:** A digraph is weakly connected, if it is connected as an undirected graph in which the direction of the edges is neglected.

**Example**

- Strongly connected
- Unilaterally connected but not strongly connected
- Weakly connected but not unilaterally connected

**Note:**
- A unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilaterally connected.
- A strongly connected digraph is both unilaterally and weakly connected.

**Problem**

Check the given graph is strongly connected, weakly connected, and unilaterally connected or not.

<table>
<thead>
<tr>
<th>Paths for vertices (A, B)</th>
<th>A-B</th>
<th>B-D-A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paths for vertices (A, D)</td>
<td>A-B-D</td>
<td>D-A</td>
</tr>
<tr>
<td>Paths for vertices (A, C)</td>
<td>A-C</td>
<td>C-B-D-A</td>
</tr>
<tr>
<td>Paths for vertices (B, C)</td>
<td>B-D-C</td>
<td>C-D</td>
</tr>
<tr>
<td>Paths for vertices (B, D)</td>
<td>B-D</td>
<td>D-A-B</td>
</tr>
<tr>
<td>Paths for vertices (C, D)</td>
<td>C-B-D</td>
<td>D-C</td>
</tr>
</tbody>
</table>

Since there is a path from each of the possible pairs of vertices of $A, B, C, D$, the given graph is strongly connected. It is both weakly and unilaterally connected.
2. Show that the given graph is unilaterally connected.

\[ \text{Path for vertices } (V_1, V_2) \quad V_1 \rightarrow V_2 \quad \text{No path from } V_2 \rightarrow V_1 \]

\[ \therefore \text{The graph is unilaterally connected.} \]

3. Determine whether each of these graphs is strongly connected, if not whether it is weakly connected.

(a) [Diagram of graph with weakly connected vertices]

(b) [Diagram of graph with weakly connected vertices]

(c) [Diagram of graph with not strongly or weakly connected]

4. If a graph has \( n \) vertices & a vertex \( v \) is connected to a vertex \( w \), then there exists a path from \( v \) to \( w \) of length not more than \( (n-1) \).

To: \( G_n; \text{ } \mid v \mid = n \).

Let \( V_1, V_2, \ldots, V_{m-1}, W \) be a path in \( G \) from \( V \) to \( W \).

By def. of the path, the vertices \( V_1, V_2, \ldots, V_{m-1}, W \) all are distinct.

As \( G \) contains only \( n \) vertices, it follows that \( m+1 \leq n \).

5. Prove that a simple graph with \( n \) vertices must be connected if it has more than \( \frac{(n-1)(n-2)}{2} \) edges.

Consider a simple graph on \( n \) vertices.

Choose \( (n-1) \) vertices \( V_1, V_2, \ldots, V_{n-1} \) of \( G \).

Clearly, maximum number of edges only \( n-1 \) can be drawn between these vertices.

\[ E = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2} \]

Thus, if we have more than \( \frac{(n-1)(n-2)}{2} \) edges, at least one edge should be drawn between the \( n \)th vertex \( V_n \) to some vertex \( V_i \) \( (1 \leq i \leq n-1) \) of \( G \).

Hence, \( G \) must be connected.

6. The minimum degree of a simple graph \( G \) of \( n \) vertices satisfies the condition \( \text{deg}(a) \geq \frac{n-1}{2} \). Then \( G \) is connected.

Theorem:

Let \( G \) be a simple graph with \( n \) vertices. Show that if \( \delta(G) > \frac{n}{2} \), then \( G \) is connected, where \( \delta(G) \) is the minimum degree of graph \( G \).
A given graph of \( n \) vertices:

\[
\text{deg}(G(n)) = \frac{n-1}{2}
\]

1. \( G \) is connected.

2. Suppose \( G \) is not connected. \( G \) is disconnected.

\( G \) has more than 1 component.

\( G \) has at least 2 components, \( G_1 \) and \( G_2 \).

Let \( G_1 \) be any component. Let \( V_1 \) be the vertex set of \( G \). Take \( v \in V_1 \). Since \( v \) is adjacent to \( v_i \),

\( V_1 \) contains at least \( \frac{p+1}{2} \) vertices. \( V_1 \) contains at least \( \frac{p+1}{2} \) vertices.

Each component of \( G \) contains at least \( \frac{p+1}{2} \) vertices.

As \( G_1 \) contains \( \frac{p+1}{2} \) vertices, \( G_2 \) contains \( \frac{p+1}{2} \) vertices.

These are \( \frac{p+1}{2} + \frac{p+1}{2} = p+1 \) vertices both in \( G_1 \) and \( G_2 \).

Which is a contradiction. \( G \) has only 1 vertex.

Our assumption is wrong.

Hence \( G \) is connected.

**Theorem:**

Let \( G \) be a simple graph with \( n \) vertices and \( k \) components. Then the number of edges \( E(G) \leq \frac{n(n-k)}{2} \).

Prove that a simple graph with \( n \) vertices \& \( k \) components can have at most \( \frac{n(n-k)}{2} \) edges.

**Proof:**

\[
E(G) \leq \frac{n(n-k)}{2}
\]

Let \( n_1, n_2, \ldots, n_k \) be the numbers of vertices in each of the \( k \) components of \( G \). Let \( n_1, n_2, \ldots, n_k \) be the numbers of vertices in each of the \( k \) components of \( G \).

Then \( n_1 + n_2 + \cdots + n_k = n = |V(G)| \).

Let \( \sum_{i=1}^{k} n_i = n \Rightarrow 0 \)

Now \( \sum_{i=1}^{k} n_i = n \Rightarrow \)

\[
\left( \sum_{i=1}^{k} n_i \right)^2 = (n_1 + n_2 + \cdots + n_k)^2 = \sum_{i=1}^{k} n_i - k = n - k
\]

Squaring on both sides:

\[
\left( \sum_{i=1}^{k} n_i \right)^2 = (n_1 + n_2 + \cdots + n_k)^2 = \sum_{i=1}^{k} n_i - k = n - k
\]

Since each \( n_i \geq 1 \) we have \( n_i > 0 \)

\[
(n_1 + n_2 + \cdots + n_k)^2 = \sum_{i=1}^{k} n_i - k = n - k
\]

And the sum of \( (n_1 - 1) \), \( (n_2 - 1) \), \( \ldots \), \( (n_k - 1) \): \( n_1 - 1 \), \( n_2 - 1 \), \( \ldots \), \( n_k - 1 \)
\[(n_1^2 - 2n_1 + 1) + (n_2^2 - 2n_2 + 1) + \cdots + (n_k^2 - 2n_k + 1) \leq n^2 + k^2 - 2nk. \]

\[\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k.\]

\[= n^2 + k^2 - k - 2nk + 2n\]

\[= n^2 + \frac{1}{2}(k^2 - 2nk + 2n)\]

\[= n^2 + \frac{1}{2}(k^2 - 2nk + 2n - k + k - 2nk - 2n + k - 2)\]

\[\leq \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k + k - 2\]

\[\Rightarrow \quad \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k + k - 2\]

Since \( G \) is a simple graph, the maximum no. of edges of \( G \) in its components is \( \frac{n_2(n_2 - 1)}{2} \). [By Thm 43]

Maximum no. of edges of \( G \) is \( \sum_{i=1}^{k} n_i^2 \leq \frac{n^2 + k^2 - k - 2nk + 2n}{2} \)

\[= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2} \leq \frac{k}{2} \leq n_i^2. \]

Using (1 & (2))

\[= \frac{1}{2} \left[ n^2 - 2nk + k^2 + 2n - k - 2 \right]\]

\[= \frac{1}{2} \left[ n^2 - 2nk + k^2 + n - k \right]\]

\[= \frac{1}{2} \left[ (n-k)^2 + (n-k) \right]\]

\[= \frac{1}{2} (n-k) (n-k+1)\]

\[= \frac{1}{2} \left( (n-k)^2 + (n-k+1) \right)\]

\[\Rightarrow \quad \text{Maximum number of edges of } G \leq \frac{(n-k)(n-k+1)}{2}\]

a) If the simple graph \( G \) has \( V \) vertices \& \( E \) edges, how many edges does \( G' \) have?

\[\text{SOL: } |G'| = E \cup (V \setminus \{a\}) = \frac{V(V-1)}{2}\]

\[|E(G)| + |E(G')| = |E(G) \cup (G \setminus \{a\})| = \frac{V(V-1)}{2}\]

\[e + |E(G')| = \frac{V(V-1)}{2} \Rightarrow |E(G)| = \frac{V(V-1)}{2} - e.\]

\[G' \text{ has } \frac{V(V-1)}{2} - e \text{ edges.}\]

b) If the simple graph \( G \) has \( V \) vertices \& \( E \) edges, then how many edges does \( G' \) have?

\[\text{SOL: } G' \text{ have } 4(4-5) - 5 = 6 \Rightarrow 5 = 1 \text{ edge.}\]
THEOREM 9:
If a graph, which may be connected or disconnected, has exactly two vertices of odd degree, there must be a path joining these vertices.

Proof:
Let $G$ be a graph with exactly two vertices $u, v$ of odd degree and all other vertices are of even degree.

Case (i) If $G$ is connected.
If graph is connected there is nothing to prove.

Case (ii) If $G$ is disconnected.
If graph is disconnected it has a number of connected components, each component itself is a graph.

By theorem 2, the number of odd vertices in a graph is even.
So the two vertices $u, v$ must belong to the same connected component $G'$ hence there must be a path joining $u, v$.

THEOREM 10:
A graph $G$ is disconnected if its vertex set $V$ can be partitioned into two non-empty disjoint subsets $V_1, V_2$ such that there exist no edge in $G$ with one end vertex in $V_1$ and the other in $V_2$.

Proof:
Let $G$ be a disconnected graph and $u$ be a vertex of $G$.
Let $V_1$ be the set of all vertices which are connected by paths since the graph $G$ is disconnected.
So the remaining vertices form a non-empty set $V_2$.
Also no vertex in $V_1$ is connected to any vertex of $V_2$.
So there is no edge in $G$ with one end vertex in $V_1$ and other in $V_2$.

Conversely:
Let the vertex set $V$ be partitioned into two subsets $V_1, V_2$ such that there is no edge joining a vertex of $V_1$ to a vertex of $V_2$.

Let $u \in V_1, v \in V_2$. There is no path between $u, v$.
For if there exist a path then there must be at least one edge whose one end vertex is in $V_1$ and the other is in $V_2$.

Which is a contradiction to our assumption.
Thus we have a pair of vertices $u, v$ such that there is no path between them.
So $G$ is disconnected.
Euler and Hamilton Paths

Königsberg Bridge Problem
Explain Königsberg bridge problem. Represent the problem by means of a graph. Does the problem have a solution?

There are two islands A and B formed by a river. They are connected to each other and to the river banks C and D by means of 7 bridges.

Problem of Königsberg Bridge is to start from any one of the 4 land areas A, B, C, D, walk across each bridge exactly once and return to the starting point.

Graphical Representation
When the situation is represented by a graph, with vertices representing the land areas and edges representing the bridges, the graph is shown in fig.

The problem is to find whether there is an Eulerian circuit or cycle (i.e., a circuit containing every edge) in the graph. Here, we cannot find an Eulerian circuit. Hence, Königsberg bridge problem has no solution.

Eulerian Path:
A path of a graph G is called an Eulerian Path if it contains each edge of the graph exactly once (is a simple path).

Eulerian Cycle / Eulerian Circuit:
A circuit or cycle of a graph G is called an Eulerian cycle if it includes each edge of G exactly once, where starting and ending vertex are same.

Eulerian Graph / Euler Graph:
Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.
When are the following graphs have Euler circuit or Euler Path?

(a) \[ \begin{array}{c}
Euler \text{ Circuit/Cycle:} \\
\text{No, exact one.}
\end{array} \]

(b) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(c) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(d) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(e) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(f) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(g) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(h) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Circuit:} \quad \text{No}
\end{array} \]

(i) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Cycle:} \quad \text{No}
\end{array} \]

(j) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Cycle:} \quad \text{No}
\end{array} \]

(k) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Cycle:} \quad \text{No}
\end{array} \]

(l) \[ \begin{array}{c}
Euler \text{ Path:} \\
Euler \text{ Cycle:} \quad \text{No}
\end{array} \]

Theorem: A connected multigraph with at least two vertices has an Euler Circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler Circuit if and only if it has exactly two vertices of odd degree.
(a) All vertices are of even degree. 

\[ \text{deg}(a) = 2 \]
\[ \text{deg}(b) = 2 \]
\[ \text{deg}(c) = 2 \]
\[ \text{deg}(d) = 2 \]
\[ \text{deg}(e) = 2 \]
\[ \text{deg}(f) = 2 \]

\( G_1 \) has an Euler circuit namely \( a - b - c - d - e - c - f - a \).

(b) All vertices are of even degree.

\[ \text{deg}(a) = 2 \]
\[ \text{deg}(b) = 2 \]
\[ \text{deg}(c) = 2 \]

\( G_2 \) has an Euler circuit namely \( a - b - c - a \).

(c) There are 6 vertices with odd degree.

\[ \text{deg}(a) = 2 \]
\[ \text{deg}(b) = 3 \]
\[ \text{deg}(c) = 2 \]
\[ \text{deg}(d) = 3 \]

\( G_3 \) has an Euler path namely \( d - a - b - c - d - b \).

(d) There are more than 2 vertices with odd degree.

\[ \text{deg}(a) = 2 \]
\[ \text{deg}(b) = 2 \]
\[ \text{deg}(c) = 3 \]
\[ \text{deg}(d) = 3 \]
\[ \text{deg}(e) = 2 \]
\[ \text{deg}(f) = 3 \]
\[ \text{deg}(g) = 3 \]

\( G_4 \) has no Euler path.

(e) \( G_5 \) has no Euler path & no Euler graph.

(f) \( G_6 \) is a Euler graph.

No Euler Circuit

\( G_7 \) does not have a Euler graph but has a Euler path \( V_1 - V_3 - V_2 - V_4 - V_1 \).

(g) \( G_8 \) is a Euler graph. 

\[ \text{deg}(V_1) = 2 \]
\[ \text{deg}(V_2) = 2 \]
\[ \text{deg}(V_3) = 2 \]
\[ \text{deg}(V_4) = 2 \]
\[ \text{deg}(V_5) = 2 \]

(h) Degree of each vertex \( \geq 4 \).

\( K_5 \) has an Eulerian Circuit.

(i) \( K_6 \) has an Eulerian Circuit.

(j) \( K_7 \) has no Eulerian Circuit.

(k) \( K_8 \) has an Eulerian Circuit.

(l) \( K_9 \) has an Eulerian Circuit.

Hence there exists an Euler path between A & B.

A - B - C - D - A - C - B.
A connected graph is Euler graph or contain Euler Cycle if each of its vertices is of even degree.

**Proof:** Necessary Part

Given $G$ is a connected graph with Euler Cycle.

**T-P:** $G$ has all the vertices of even degree.

Since $G$ is a connected graph with Euler Cycle.

Let $C$ be Eulerian Cycle of $G$ with origin and vertex as $v$.

Each time a vertex $v$ occurs as internal vertex of $C$, then two of the edges incident with $v$ are accounted for.

We get, for internal vertex $u \in V(G)$,

$d(u) = 2 \times \text{No. of times } v \text{ occur inside the Eulerian Circuit C}$

$d(v) = \text{Even degree.}$

And since an Eulerian Circuit $C$ contains every edge of $G$ and starts and ends at $u$.

$d(u) = 2 + 2 \times \text{No. of times } u \text{ occur inside } C \Rightarrow \text{Even degree.}$

$G$ has all the vertices of even degree.

Conversely: Sufficient Part

Assume: $G$ is connected graph in which each vertex is of even degree.

**T-P:** $G$ is Eulerian or Euler Graph.

**W.T.P:** $G$ contains a Eulerian Cycle / Euler Circuit.

An Eulerian Circuit is obtained by $G$ by procedure in which circuits are spliced until to get the circuit.

- Start from any vertex $v$ and traverse distinct edges of $G$ until we return to $v$. This is certainly possible since each vertex is of even degree. Let $C_1$ be the circuit thus obtained.

- If this circuit $C_1$ contains all the edges of graph $G$, then $C_1$ is an Euler Cycle.

- Otherwise, delete all the edges of this circuit $C_1$ and all vertices of degree 0 from $G$ to obtain the connected subgraph $H$, in which each vertex is also even.

  Further more, since $G$ is connected, there is a vertex $u$. Now start from $u$ and obtain a circuit $C_2$ by traversing distinct edges of the subgraph $H$. 


We note that the two circuits have no common edges even though they may have common vertices. If $v = u$, the two circuits can be joined together to form an enlarged circuit $C_3$.

If $u \neq v$ are distinct, let $P$ and $Q$ be the two distinct simple paths between $u$ and $v$. Consisting of edges from $G$, then $P, Q \subseteq C_2$ are spliced together to form a new circuit $C_3$. If this enlarged circuit has all the edges of $G$, we conclude that it is a Eulerian.

Otherwise, we continue until we obtain a circuit that has all the edges of $G$.

Hence the proof.

**Theorem 12**: A connected graph has an Euler path if and only if it has exactly two vertices of odd degree.

**Proof**: Necessary Part

Assume $G$ has an Euler path.

1. $G$ contains at most two vertices of odd degree.

2. $G$ has an Euler path by previous theorem (State 15), each vertex other than the origin and terminus of this path has even degree. Therefore, $G$ contains at most two vertices of odd degree.

Conversely: Sufficient Part

Assume $G$ is a connected graph with at most two vertices of odd degree.

1. If $G$ has no such vertices, then by (previous theorem) $G$ has a closed Euler cycle.

Otherwise, $G$ has exactly two vertices $u$ and $v$ of odd degree. In this case, let $G' = G + uv$, obtained from $G$ by the addition of a new edge joining $u$ and $v$. Clearly, each vertex of $G'$ has even degree, so by previous theorem $G'$ has an Euler path.
**HAMILTONIAN GRAPH**

**HAMILTONIAN PATH:**
A path of a graph $G$ is called a Hamiltonian path if it includes each vertex of $G$ exactly once.

**HAMILTONIAN CIRCUIT OR CYCLE:**
A circuit (cycle) of a graph $G$ is called a Hamiltonian circuit (cycle) if it includes each vertex of $G$ exactly once, except the starting and ending vertices.

**HAMILTONIAN GRAPH:**
Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.

**PROPERTIES OF HAMILTONIAN GRAPH**

1. A Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian cycle.
2. By deleting any one edge from Hamiltonian cycle, we can get Hamiltonian path.
3. A graph may contain more than one Hamiltonian cycle.
4. A complete graph $K_n$, will always have a Hamiltonian cycle, when $n \geq 3$. 
**Hamiltonian Path**

[A path in which all vertices appear exactly once]

- **Hamiltonian Path:**
  - $V_1 - V_2 - V_3 - V_4 - V_3$

**Hamiltonian Cycle / Graph**

[All vertices appear exactly once (not all the edges), starting and ending vertex are same]

- **Cycle:** $V_5 - V_2 - V_3 - V_4 - V_5$

**Remark:**

A graph with a vertex of degree one cannot have a Hamiltonian Cycle.

There is no Hamiltonian cycle / graph.

Give an example of a graph which is

- (a) Eulerian but not Hamiltonian
- (b) Hamiltonian but not Eulerian
- (c) Both Eulerian and Hamiltonian
- (d) Not Eulerian and not Hamiltonian

**Solution:**

(a) Example of Eulerian graph but not a Hamiltonian Graph.

**Euler Cycle → Eulerian Graph**

- $A - B - C - D - B - E - A$
  - [degree of all vertices of $G_1$ is even]

No Hamiltonian Cycle → No Hamiltonian Graph

- (G2 vertex B is repeated twice)

Hence, $G_1$ is Eulerian but not Hamiltonian Graph.
Hamiltonian Cycle → Hamiltonian Graph
A → B → C → D → E → A

Eulerian Graph (Cycle): No
All vertices should be of even degree. Here, deg(A) = 3, deg(B) = 3.

→ G₂ is Hamiltonian graph but not Eulerian graph.

Example of both Eulerian and Hamiltonian.

G₃ is both Eulerian graph & Hamiltonian graph.

Example of Neither Eulerian nor Hamiltonian.

G₄ is neither Eulerian graph nor Hamiltonian graph.

Theorem: A complete graph with n vertices, Kₙ, has a Hamiltonian circuit when n ≥ 3. (a) The complete graph Kₙ contains \(\binom{n \cdot (n-1)}{2}\) different Hamiltonian cycles.

Proof: Gₙ: Complete graph Kₙ.

① P: No. of different Hamiltonian cycles = \(\frac{(n-1)!}{2}\)

Complete graph Kₙ on vertices → Starting from any vertex (n-1) edges to choose from the first vertex, (n-2) from the second vertex, (n-3) from the third vertex, and so on. All these are mutually exclusive, the possible no. of choices is \(\frac{(n-1)!}{2}\).

Each choice gives a Hamiltonian circuit/cycle, counts from each end. Hence there are \(\frac{(n-1)!}{2}\) Hamiltonian cycle in a graph.
**Hamiltonian Graph Theorems**

---

**Ore's Theorem**: If \( G \) is a simple graph with number of vertices \( n \geq 3 \) and \( \deg(u) + \deg(v) \geq n \) for every pair of non-adjacent vertices \( u \) and \( v \), then \( G \) is Hamiltonian.

**Proof**: Suppose \( G \) is Hamiltonian. We shall prove the theorem by contradiction.
Let \( u \) and \( v \) be non-adjacent vertices in \( G \).

By choice of \( G \), \( G + uv \) is Hamiltonian. Simple graph.

Thus there is a Hamiltonian path \( P = u, v \) in \( G + uv \).

Define: \( s = s(u, v) \) adjacent to vertex \( u \), \( w = w(u, v) \) adjacent to vertex \( v \).

Clearly, \( 1 + \deg(u) + 1 + \deg(v) \geq n \).

If \( G + uv \) contained some vertex \( v \), then \( G \) would have the Hamiltonian cycle \( u, v, u, v \) which is contradiction.

Also, \( \deg(u) + \deg(v) = 1 + 1 + 1 + 1 = 4 \geq n \).

But it is a contradiction to our hypothesis. Hence our assumption is wrong. So, \( G \) is Hamiltonian.

---

**Dirac's Theorem**: If \( G \) is a connected simple graph with \( n \) vertices, where \( n \geq 3 \), then \( G \) is Hamiltonian if the degree of each vertex is at least \( \frac{n}{2} \).

**Proof**: \( G_n : G \) is connected simple path, \( n \geq 3 \).

\[ \deg(u) + \deg(v) = \frac{n}{2} + \frac{n}{2} = n \]
THEOREM 14 Hold

Prove the Converse, Part of THEOREM No. 14.

Hence the Proof.
**Theorem.**

A simple graph \( G \) is bipartite if it is possible to assign one of two different colors to each vertex of the graph \( G \), so that no two adjacent vertices are assigned the same color.

**Proof:**

First, assume \( G = (V, E) \) is a bipartite graph. Then \( V = V_1 \cup V_2 \) where \( V_1 \) and \( V_2 \) are disjoint sets and every edge in \( E \) connects a vertex in \( V_1 \) to a vertex in \( V_2 \). If we assign one color to each vertex in \( V_1 \) and a second color to each vertex in \( V_2 \), then no two adjacent vertices are assigned the same color.

Now suppose that it is possible to assign colors to the vertices of the graph using just two colors \( B \) so that no two adjacent vertices are assigned the same color and let \( V_1 \) be the set of vertices which are assigned the other color. Then \( V_1 \) and \( V_2 \) are sets of vertices assigned the other color \( B \). Then \( V_1 \) and \( V_2 \) are disjoint sets and \( V = V_1 \cup V_2 \).

Furthermore, every edge connects a vertex in \( V_1 \) and a vertex in \( V_2 \) because no two adjacent vertices are either both in \( V_1 \) or \( V_2 \).

Consequently, \( G \) is bipartite.

---

**Drawing Graphs from Given Conditions**

Draw the graph with 5 vertices \( A, B, C, D, E \) such that:

- \( \deg(A) = 3 \), \( B \) is odd.
- \( \deg(C) = 2 \) and \( D \) and \( E \) are adjacent.

Solution:

On the vertices are \( A, B, C, D, E \) such that:

- \( \deg(A) = 3 \)
- \( \deg(C) = 2 \)
- \( D \) and \( E \) are adjacent.
1. Complete graph $K_5$ with vertices $A, B, C, D, E$.

2. Draw all complete subgraphs of $K_5$ with 4 vertices.

3. To find: Subgraphs of $K_5$ with 4 vertices.

4. When a vertex is omitted the edges adjacent with it will also be omitted.

Thus we have 5 subgraphs of $K_5$ with vertices 4.

5. Determine which of the following graphs are bipartite and which are not. If a graph is bipartite, state if it is completely bipartite.

6. Solution: A simple graph $G$ is bipartite if the vertex set can be partitioned into non-empty subsets $V_1 \cup V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2$.

(a) In $G_1$, since the vertices $D, E, F$ are not connected we can consider them as $V_1 = \{D, E, F\}$ and $V_2 = \{A, B, C\}$.

   But then $A \in V_1$ and $B \in V_2$ are joined.

   Hence the edges $AB, BC$ have both ends in $V_2$.

   Graph $G_1$ is not bipartite.

(b) In $G_2$, if we take $V_1 = \{A, C\}$ and $V_2 = \{B, D, E\}$ it is seen that no two vertices of $V_1$ are joined and no two vertices of $V_2$ are joined.

   Every edge of $G_2$ has one end in $V_1$ and other end in $V_2$.

   $G_2$ is bipartite.

$G_2$ is completely bipartite (i.e., every vertex of $V_1$ is joined to every vertex of $V_2$).
In $G_3$, $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$. So clearly $G_3$ is bipartite with $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$. But it is not completely bipartite, because $A$ is not joined to $E$ and $C$ is not joined to $F$.

6. Which of the following graphs are Eulerian?

a. The graph of the octahedron.

Each vertex in the graph of the octahedron is of degree 4, which is even. Hence the graph of the octahedron is an Euler graph.

b. The Peterson graph.

It is clear that the Peterson graph is not an Euler graph since each vertex of this graph is of degree 3, which is not even.
1. Define Graph. [Alu M/1 '10]
   Sol: A graph \( G = (V, E, \phi) \) consists of a non-empty set \( V = \{v_1, v_2, \ldots\} \) called the set of nodes [points/vertices] of the graph, \( E = \{e_1, e_2, \ldots\} \) is said to be the set of edges of the graph and \( \phi \) is mapping from the set of edges \( E \) to set of ordered or unordered pairs of elements of \( V \).

2. Define Pseudograph. [AlM '11]
   Sol: Graphs that include loops and possibly multiple edges connecting the same pair of vertices are sometimes called Pseudograph.

3. Define a Complete graph with example. [AlM '11]
   Sol: A graph \( G \) is said to be complete if every vertex in \( G \) is connected to every other vertex in \( G \). Eg:
   ![Complete Graph Example]
   Thus a complete graph in must be connected.

4. Define a Connected graph and disconnected graph with examples. [AlM '11]
   Sol: Connected Graph: A graph \( G \) is said to be connected if every pair of vertices in \( G \) are joined by a path.
   Disconnected Graph: If \( G \) is not connected then \( G \) is called a disconnected graph.

5. Define a regular graph. Can a complete graph be a regular graph? [HRI '12]
   Sol: A graph in which all vertices are of equal degree is called a Regular Graph. A complete graph is a regular graph.

6. Define pendant vertex in a graph. [Alu '05, '07]
   Sol: A vertex in a graph is called pendant vertex iff one edge is incident with it i.e., degree of the vertex is 1.
1. When a simple graph is bipartite graph. Give Eq. \[ \text{Eq. (10.11)} \]

Sol: A simple graph \( G \) is called bipartite graph if its vertex set \( V \) can be partitioned into 2 disjoint, non-empty sets \( V_1 \), \( V_2 \) st every edge in \( G \) connects a vertex in \( V_1 \) and vertex in \( V_2 \).

\[ V_1 = \{v_1, v_3, v_5\} \quad \text{and} \quad V_2 = \{v_2, v_4, v_6\} \]

Here \( G = V_1 \cup V_2 \) \& \( V_1 \cap V_2 = \emptyset \)

\[ C_6 \text{ is Bipartite graph.} \]

2. Draw a Complete Bipartite Graph of \( K_{2,3} \) & \( K_{3,2} \).

Sol: Complete Bipartite Graph:

A complete bipartite graph is a bipartite graph in which every vertex of \( M \) is adjacent to every vertex of \( N \). The complete bipartite graphs that may be partitioned into sets \( M, N \) such that \( |M| = m \) & \( |N| = n \) are denoted by \( K_{m,n} \).

\[ K_{2,3} : \quad K_{3,2} : \]

3. Define isomorphism of 2 graphs. [ALP] \[ \text{[ML3/14]} \quad \text{[ML3/10]} \]

Sol: Two graphs \( G \) & \( G' \) are isomorphic if there is a function \( f : V(G) \rightarrow V(G') \) from the vertices of \( G \) \( \cup \) \( f \) is 1-1 \( \cup \) \( f \) is onto \( \cup \) \( f \) preserves adjacency.

4. State the Handshaking theorem. [HIO '12].

Sol: Handshaking Theorem: The sum of degrees of the vertices of an undirected graph \( G \) is twice the number of edges in \( G \).

5. What is the number of edges in \( k \)-regular graph.

Sol: If \( G \) has \( n \) vertices and regular of degree \( k \) then

Total no. of edges in \( k \) \[ [\text{regular graph}] = \frac{kn}{2} \]

6. What is the number of edges in Complete graph, \( K_n \)?

Sol: \( K_n \) - Complete graph is also called \( (n-1) \) Regular graph.

Total no. of edges in \( K_n = \frac{nC_2}{2} = nC_2 = \frac{n(n-1)}{2} \)

7. Find the number of edges and degree of each vertex in complete graph \( K_5 \).

Sol: \( K_5 \) - Complete graph with 5 vertices.

The no. of edges = \( C_5^2 = 5 \times 4 = 10 \).

The degree of each vertex = \( n-1 = 4-1 = 4 \).
15. What is the maximum no. of edges of a simple graph with 10 vertices?
Sol: The maximum no. of edges = \( nC_2 = 10C_2 = 45 \).

15. Can a simple graph with 8 vertices have 40 edges?
Sol: The maximum number of edges = \( nC_2 = 8C_2 = 28 \)
So it is not possible to have 40 edges.

16. Does there exist a simple graph of order 4 and size 7?
Sol: Order of \( G = n = 4 \); Size of \( G = |E| = 7 \)
W.K.T Maximum number of edges = \( nC_2 = 4C_2 = 6 \).
\[ |E| = 7 > 6 \] there cannot be a graph with order 4 and size 7.

17. Does there exist a graph with 13 vertices each of degree 3.
Sol: No. of vertices = \( |V| = 13 \); \( \deg (V) = 3 \).
Let, no. of edges = \( |E| = m \)
By Handshaking theorem, \( \sum \deg (V) = 2|E| = 2m \)
\[ 3 + 3 + \ldots + 3 = 2m \Rightarrow 13 \times 3 = 2m \Rightarrow m = \frac{39}{2} \] is not an int
So we cannot have a graph with 13 vertices and degree of each vertex = 3.

18. A regular graph \( G \) has 10 edges and degree of any vertex is 5, find the no. of vertices.
Sol: No. of edges = \( |E| = 10 \)
Degree of each vertex = 5 \[ \Rightarrow G \ is \ regular \ graph \]
Let no. of vertex = \( n \).
By Handshaking theorem, \[ \frac{n}{2} \deg (V) = 2|E| = 2 \times 10 \]
\[ \frac{5 + 5 + \ldots + 5}{n \text{ times}} = 20 \]
\[ 5 \times n = 20 \]
\[ n = 4 \]

19. Does there exist a simple graph with the degree sequence \( (5, 5, 5, 5, 3) \)?
Sol: Degree sequence : \( (5, 5, 5, 5, 3) \)
\[ \sum \deg (V) = 5 + 5 + 5 + 5 + 3 = 23 \] is not even
And also the no. of odd degree is not even.
So, such a graph is not possible.
20. How many edges are there in a graph with 10 vertices each of degree 5?
Sol: |V| = 10, degree of each vertex = 5
By Handshaking theorem \( \frac{1}{2} \sum d(v_i) = 2e \).
\[
5 + 5 + \ldots + 5 = 2e \\
[10 \text{times}]
\]
\[
5 \times 10 = 2e \Rightarrow e = 25
\]

There are 25 edges.

21. If the graph has \( n \) vertices and a vertex \( v' \) is connected to a vertex \( w' \), then there exists a path from \( v' \) to \( w' \) of length not more than \( (n-1) \).
Sol: Let \( v, v_1, v_2, \ldots, v_{m-1}, w \) be a path in \( G \) from \( v \) to \( w \).
By definition of path, the vertices \( v, v_1, v_2, \ldots, v_{m-1}, w \) are all distinct.
As \( G \) contains only \( n \) vertices, it follows that \( m \leq n \).
[Def: PATH: A path in a graph is a sequence \( v_1, v_2, \ldots, v_k \) of vertices each adjacent to the next.]

22. If the simple graph \( G \) has \( v \) vertices and \( e \) edges, how many edges does \( G' \) have?
Sol: \( |E(G \cup G')| = \frac{v(v-1)}{2} \)
\[
|E(G)| + |E(G')| = \frac{v(v-1)}{2}
\]
\[
e + |E(G')| = \frac{v(v-1)}{2}
\]
\[
|E(G')| = \frac{v(v-1) - e}{2}
\]
\( G' \) has \( \frac{v(v-1) - e}{2} \) edges.

23. Draw the graph represented by the given adjacency matrix.
\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Sol:

24. Obtain an adjacency matrix to represent the pseudograph shown below.
Sol:
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
29. Find the matrix of the following graphs. Hence find the degree of the vertices $v_1, v_2, v_3$ and $v_5$.

Sol:

$$A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \\ v_5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$d(v_1) =$ Sum of the entries in the $1^{st}$ row $= 3 + 1 + 1 = 5$

$d(v_2) =$ Sum of the entries in the $2^{nd}$ row $= 1 + 1 + 1 = 3$

$d(v_5) =$ Sum of the entries in the $5^{th}$ row $= 0 + 1 + 1 + 1 = 2$

26. Find Incidence matrix of the graph:

Sol:

$$B = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

28. Obtain adjacency matrix to represent the pseudograph shown below.

Sol:

$$A = \begin{bmatrix} a & b & c & d \\ b & 0 & 3 & 0 \\ c & 0 & 1 & 2 \\ d & 2 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c & d \\ b & 1 & 0 & 2 \\ c & 0 & 1 & 1 \\ d & 2 & 1 & 0 \end{bmatrix}$$
80. Draw the graph represented by the given adjacency matrix.

\[
V = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

Sol:

\[
\begin{align*}
V_1 & \quad V_2 & \quad V_3 & \quad V_4 \\
\end{align*}
\]

81. Write the adjacency matrix of the digraph \( G = \{ (V_1, V_2), (V_1, V_3), (V_2, V_4), (V_3, V_1), (V_2, V_3), (V_3, V_4), (V_4, V_1), (V_4, V_2), (V_3, V_4) \} \). Also draw the graph.

Sol:

\[
A(G) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

82. Draw the graph \( G \) whose incidence matrix is given below.

Sol:

\[
I = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

83. Find the incidence matrix of a Digraph.

Sol: Incidence matrix of directed graph.

\[
B = \begin{bmatrix}
1 & 0 & -1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

84. Find the various degrees of

<table>
<thead>
<tr>
<th>In-degree</th>
<th>Out-degree</th>
<th>Total degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{deg}^{-}(a) = 3 )</td>
<td>( \text{deg}^{+}(a) = 1 )</td>
<td>( \text{d}(a) = 4 )</td>
</tr>
<tr>
<td>( \text{deg}^{-}(c) = 2 )</td>
<td>( \text{deg}^{+}(c) = 1 )</td>
<td>( \text{d}(c) = 3 )</td>
</tr>
<tr>
<td>( \text{deg}^{-}(d) = 1 )</td>
<td>( \text{deg}^{+}(d) = 3 )</td>
<td>( \text{d}(d) = 5 )</td>
</tr>
</tbody>
</table>
35. **Strongly Connected**
   A simple digraph is said to be strongly connected if for any pair of nodes of the graph both the nodes of the pair are reachable from one another.

**Weakly Connected**
We call a digraph is weakly connected if it is connected as an undirected graph in which the direction of the edges is neglected.

36. Is the directed graph given below strongly connected why or why not?
   **Sol:** No the directed graph given is not strongly connected, since the directed graph has no path from u to v and from v to u.

37. **Find the no. of connected simple graph with 4 vertices**
   **Sol:**
   There are 6 simple graph (non-isomorphic).

38. **How many non-isomorphic connected simple graphs are there with 3 vertices?**
   **Sol:** There are only 2 connected graph with 3 vertices.

39. **Define Euler graph?**
   **Sol:** A connected graph with an Euler Circuit is called Euler graph.

**Euler Circuit**
A circuit of a graph G is said to be an Eulerian Circuit if it includes each edge of G exactly once starting and ending points are same.

**Eg:**

40. **State the necessary and sufficient conditions for the existence of an Eulerian Path in a Connected Graph.**
   **Sol:** A connected graph has an Eulerian Path not an
Euler circuit if and only if it has exactly two vertices of odd degree.

(3) State the necessary and sufficient conditions for the existence of an Eulerian graph in a connected graph.
Sol: A connected graph G is Eulerian graph if and only if every vertex of G is of even degree.

(4) Give an example of a non-Eulerian graph which is Hamiltonian.

Sol: Hamiltonian graph → Hamiltonian cycle
\[ V_1, V_2, V_3, V_4, V_5, V_6 \]
Non-Eulerian → Because the degree of each vertex is not even.

(4) Give an example of a simple graph G in which is Eulerian but not Hamiltonian.

Sol: Eulerian/Euler graph:
Every vertex is of even degree so has a Euler circuit.
Non-Hamiltonian:
Every circuit containing every vertex contains a vertex twice. For e.g: \[ V_1, V_2, V_3, V_4, V_5, V_6, V_3, V_1, V_2 \]