PARTIAL ORDERED SET [POSET]

1. Show that \((\mathbb{N}, \leq)\) is a partially ordered set where \(\mathbb{N}\) is set of all positive integers and \(\leq\) is defined by \(m \leq n\) if \(n - m\) is a non-negative integer. [MJD 2019]

2. Draw the Hasse diagram for (i) \(P_1 = \{1, 2, 3, 6, 12\}\)
(ii) \(P_2 = \{1, 2, 3, 4, 6, 12\}\) and \(\leq\) is a relation such that \(x \leq y\) iff \(x|y\). [IL 2011]

3. Draw the Hasse diagram representing the partial ordering \((A, P)\); \(A = \{0, 1, 2, 3\}\). Find the maximal, minimal, greatest, and least elements of the poset. [MJD 2012]

LATTICES

1. Show that in a lattice, if \(a \leq b\) and \(c \leq d\) then \(a \land c \leq b \land d\) and \(a \lor c \leq b \lor d\). [MLJ 2014]

2. In a distributive lattice prove that \(a \land (b \lor c) = (a \land b) \lor (a \land c)\). [MLJ 2014]

3. Let \(L\) be lattice, where \(a \land b = \text{glb}(a, b)\) & \(a \lor b = \text{lub}(a, b)\)
for all \(a, b \in L\). Then both binary operation \(\land\) and \(\lor\) defined as in \(L\) satisfies commutative law, associative law, absorption law and idempotent law. [MLJ 2013]

4. Show that every non-empty subset of a lattice has a least upper bound and a greatest lower bound. [MJD 2012]

5. In a distributive lattice \(L, \lor, \land\) if an element \(a \in L\) complement then it is unique. [CHD 2012]

6. Show that every chain is a lattice. [CHD 2012]

7. Prove that every chain is a distributive lattice. [MLJ 2013][MJD 2012]

8. Prove that every distributive lattice is modular. Is the converse true? Justify your claim. [MLJ 2013]

9. Show that in a distributive and complemented lattice \(a \leq b\) \(\iff a \land b = 0\) \(\iff a' \lor b = 1\) \(\iff b' = a'\). [MLJ 13]

10. Show that in a lattice if \(a \leq b \leq c\) then \(\bar{a} \lor c = b \lor c\). [MJD 12]

11. Show that the direct product of any two distributive lattices is a distributive lattice. [MLJ 12]

12. If \((P, \leq)\) is the power set of a set \(S\), and \(\cup, \cap\) are taken as join and meet, prove that \((P, \leq)\) is a lattice. Also prove the modular inequality of a lattice \((L, \leq)\) for any \(a, b, c \in L\); \(a \leq c \iff a \lor (b \land c) \leq (a \lor b) \land c\). [2011]
Boolean Algebra

1. Show that a complemented, distributive lattice is a Boolean algebra. [AIM 83]
2. In a Boolean Algebra, \( P(T(\alpha \land \beta)) = \alpha \lor \beta \). [AIM 94]
3. The De Morgan's laws hold in a Boolean Algebra: \( \neg (X \lor Y) = \neg X \land \neg Y \). [AIM 84]
4. In any Boolean Algebra, \( \neg \alpha \land \neg \beta = 0 \) if \( \alpha = \beta \). [AIM 97]
5. In any Boolean Algebra, if the following statement are equivalent:
   (a) \( \alpha \lor \beta = \beta \)
   (b) \( \alpha \land \beta = \alpha \)
   (c) \( \alpha \land \beta = 0 \)
   (d) \( \alpha \land \beta = \beta \). [AIM 115]
6. In a Boolean algebra, \( P(T(\alpha \land \beta)) = \alpha + \beta \). [AIM 126]
7. Simplify the Boolean expression \( \alpha \lor \beta \cdot \gamma + \alpha \land \beta \cdot \gamma + \alpha \land \beta \cdot \gamma \).
   Using Boolean algebra identities. [AIM 128]
8. Show that a lattice homomorphism on a Boolean algebra which preserves 0 & 1 is a Boolean homomorphism. [AIM 113]
9. Let \( B \) be a finite Boolean algebra and let \( A \) be the set of all atoms of \( B \). Prove that the Boolean Algebra \( P(A) \) (where \( P(A) \) is the power set of \( A \)). [AIM 132]
10. Prove that \( D_{100} \), the set of all positive divisors of a positive integer 100, is a Boolean algebra. Find all its subalgebras. [AIM 117]
11. If \( \alpha, \beta \in \{0, 1\} \), and \( \alpha \land \beta = \text{lcm}(\alpha, \beta) \), then show that \( \{0, 1, 1, 0, 0\} \) is a Boolean Algebra. [AIM 17]
Cartesian Product of sets:

Let $A$ and $B$ be non-empty sets. The set of all ordered pairs $\langle a, b \rangle$ from $A \times B$ is defined as the Cartesian Product of $A \times B$ denoted by $AXB$. Thus $AXB = \{\langle a, b \rangle \mid a \in A, b \in B\}$.

For example:

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$AXB = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B^A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

Relation from $A \times B$: $A \times B = (a, b) \in A \times B$

Let $A \times B$ be non-empty sets. A relation $R$ from $A \times B$ is a subset of $AXB$ such that $R \subseteq A \times B$.

$R(a, b) \Rightarrow a$ is related to $b$. (or) $(a, b) \in R$

Relation on $A$:

A relation $R$ from $A \times A$ is said to be a relation on $A$.

Example:

If $A = \{1, 2, 3, 4\}$ and $R = \{(x, y) \mid x \text{ divides } y \}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Properties of Relations on $A$:

Let $R$ be a relation on $A$.

1. Reflexive: If $R$ is reflexive, then $\forall a \in A$.
2. Symmetric: If $R$ is symmetric, then whenever $aRb$ then $bRa$.
3. Transitive: If $R$ is transitive, then whenever $aRb$ and $bRc$ then $aRc$.
4. Antisymmetric: If $R$ is antisymmetric, then whenever $aRb$ and $bRa$ then $a = b$.

Note:

1. No. of different relations from $A$ (with $m$ elements) to $B$ (with $n$ elements) = $2^{mn}$
2. No. of different relations on $A$ (with $n$ elements) = $2^{n^2}$
Example:
1. The relation \( \leq \) is reflexive.
2. The relation "Inclusion" in the collection of the subsets of a universal set is antisymmetric.
3. Let \( A \) be the set of all straight lines in a plane. The relation \( R \) "\( x \) is parallel to \( y \)" is reflexive, symmetric & transitive.

**Equivalence Relation:**
A relation \( R \) on a set \( A \) is called an equivalence relation if it is reflexive, symmetric & transitive.

Eg: Set of all lines \( R \) is "\( x \) is parallel to \( y \)."

**Partial Order Relation (P.O.R):**
A relation \( R \) on a set \( A \) is called partial order relation if \( R \) is reflexive, antisymmetric and transitive.

**Examples:**
Consider any 3 sets: \( A, B, C \). 

- **Reflexive:** \( A \subseteq A \) is reflexive 
  (Every set is a subset of itself)
- **Antisymmetric:** \( A \subset B \subset A \Rightarrow A = B \)
- **Transitive:** \( A \subset B \subset C \Rightarrow A \subset C \)

**Partial Order Set or Poset:**
A set \( P \) together with a partial ordering \( \leq \) is called a Poset.

Eg: \( \langle A, \leq \rangle \) is a Poset.

\[ R = \{ (a,b) | a \text{ divides } b \} \]

**Reflexive:** \( a \text{ divides } a \Rightarrow (a,a) \in R \) is reflexive.

**Antisymmetric:** \( a \text{ divides } b \) \( \Rightarrow b = ax \Rightarrow a = by = aby \Rightarrow x = y \)

**Transitive:** \( a \text{ divides } b \) \( \Rightarrow b = ax \) \( \Rightarrow c = by = a(xy) \Rightarrow c = a(xy) \)

\[ (a,b) \in R \Rightarrow a \leq b \]

\[ (a,b) \in R \Rightarrow b \leq c \Rightarrow (a,c) \in R \]

1. \( (a,b, c) \) is a Poset.

2. \( (a,b, c) \) is a Poset.

3. \( (a,b, c) \) is a Poset.

4. \( (a,b, c) \) is a Poset.

5. \( (a,b, c) \) is a Poset.

6. \( (a,b, c) \) is a Poset.

7. \( (a,b, c) \) is a Poset.

8. \( (a,b, c) \) is a Poset.

9. \( (a,b, c) \) is a Poset.

10. \( (a,b, c) \) is a Poset.

11. \( (a,b, c) \) is a Poset.

12. \( (a,b, c) \) is a Poset.

13. \( (a,b, c) \) is a Poset.

14. \( (a,b, c) \) is a Poset.

15. \( (a,b, c) \) is a Poset.

16. \( (a,b, c) \) is a Poset.

17. \( (a,b, c) \) is a Poset.

18. \( (a,b, c) \) is a Poset.

19. \( (a,b, c) \) is a Poset.

20. \( (a,b, c) \) is a Poset.

21. \( (a,b, c) \) is a Poset.

22. \( (a,b, c) \) is a Poset.

23. \( (a,b, c) \) is a Poset.

24. \( (a,b, c) \) is a Poset.

25. \( (a,b, c) \) is a Poset.

26. \( (a,b, c) \) is a Poset.

27. \( (a,b, c) \) is a Poset.

28. \( (a,b, c) \) is a Poset.

29. \( (a,b, c) \) is a Poset.

30. \( (a,b, c) \) is a Poset.
Let \( R \) be a relation on a set \( A \). Then define \( R^1 = \{(a,b) \in A 	imes A \mid (a,b) \in R \} \) and \( R^2 = \{(a,b) \in A 	imes A \mid (a,c) \in R \land (c,b) \in R \} \).

Prove that if \( (A, R) \) is a poset, then \( (A, R^1) \) is also a poset.

So:\( A \) must be a finite set.

**Proposition:** \( (A, R) \) is a partially ordered relation on \( A \).

\[ R^1 = \{(a,b) \in A 	imes A \mid (a,b) \in R \} \]

**Conclusion:** \( (A, R^1) \) is a partially ordered set.

**Reflexive:** \( aRa \Rightarrow aRa \) is reflexive.

**Antisymmetric:** \( aRb \Rightarrow bRa \Rightarrow a = b \)

**Transitive:** \( aRb \Rightarrow bRc \Rightarrow aRc \)

**Comparable Property:**

In a poset, for any two elements \( a, b \), either \( a \leq b \) or \( b \leq a \). It is called the Comparable Property. Otherwise, it is called an Incomparable Property.

**Totally Ordered Set or Linearly Ordered Set or Chain:**

A partially ordered set \( (P, \leq) \) is said to be totally ordered if any two elements are comparable.

**Example:**

1. \( aRb \) if \( a \leq b \) is a total order.
2. \( aRb \) if \( a \neq b \) is not a total order.
3. Given elements 283, neither 2/3 nor 3/2, so 283 are not comparable.
**HASSE DIAGRAM** (8M).

**Problems**

1. If \( A = \{a, b, c\} \)
   - The elements of \( A \) are represented by *a, b, c*.
   - If \( a \leq b \) then there is a line segment from \( a \) to \( b \).
   - If \( a \leq b \) and \( b \leq c \) then \( a \leq c \).
   - There is no line segment from \( a \) to \( c \).
   - But there is a line segment from \( a \) to \( b \) and from \( b \) to \( c \).

**Poset \((P, \leq)\)**

<table>
<thead>
<tr>
<th>Poset</th>
<th>Relation ((\leq))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([2, 1])</td>
<td>Divides</td>
</tr>
<tr>
<td>([R, \leq])</td>
<td>Less than or equal</td>
</tr>
<tr>
<td>([R, &gt;])</td>
<td>Greater than or equal</td>
</tr>
<tr>
<td>([P, X])</td>
<td>Set Inclusion</td>
</tr>
</tbody>
</table>

**Immediate Successors**

- If \( x \leq y \) and there is no element \( z \in P \) between \( x \) and \( y \),

**Immediate Successors**

- \( 1 \leq 2, 2 \leq 3, 3 \leq 4 \).

The Hasse diagram resembles a chain.

2. Let \( A \) be a given finite set and \( \mathcal{P}(A) \) its power set. Let \( \leq \) be the inclusion relation on the elements of \( \mathcal{P}(A) \). Draw Hasse diagram of \((\mathcal{P}(A), \leq)\) for \( A = \{a, b, c\} \).

- \( \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \)
- Immediate Successors:
  - \( \emptyset \leq \{a\}, \{a\} \leq \{a, b\}, \{a\} \leq \{a, b, c\} \)

3. Let \( D_{36} \) denote the set of divisors of 36. Draw the Hasse diagram for poset \((D_{36}, \leq)\).

- \( D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \)
- The immediate successors:
  - \( 1 \leq 2, 1 \leq 3, 2 \leq 4, 2 \leq 6, 3 \leq 6, 3 \leq 9, 4 \leq 12, 6 \leq 18, 6 \leq 36, 9 \leq 18, 12 \leq 36 \).
SPECIAL ELEMENTS OF A POSET

LEAST ELEMENT IN P.

Let \((P, \leq)\) be a Poset. An element \(a \in P\) is called the least element in \(P\) if \(a \leq x\) for all \(x \in P\).

GREATEST ELEMENT IN P.

An element \(b \in P\) is called the greatest element in \(P\) if \(x \leq b\) for all \(x \in P\).

Note: The least elt. is called the '0' element and the greatest elt. is '1'.

UPPER BOUND OF A

Let \((P, \leq)\) be a Poset and \(A \subseteq P\) be any non-empty subset of \(P\). An element \(a \in P\) is an upper bound of \(A\) if \(a \geq x\) for all \(x \in A\).

LOWER BOUND OF A

An element \(b \in P\) is said to be a lower bound for \(A\) if \(b \leq x\) for all \(x \in A\).

LEAST UPPER BOUND (LUB)

Let \((P, \leq)\) be a Poset \& \(A \subseteq P\). An element \(a \in P\) is said to be LUB or supremum of \(A\) if:

1) \(a\) is the upper bound of \(A\).
2) \(a\) is the least upper bound of \(A\), where \(c\) is any other upper bound of \(A\).

GREATEST LOWER BOUND (GLB)

Let \((P, \leq)\) be a Poset \& \(A \subseteq P\). An element \(b \in P\) is said to be GLB or infimum (inf) of \(A\) if:

1) \(b\) is the lower bound of \(A\).
2) \(b\) is the greatest lower bound of \(A\), where \(d\) is any other lower bound of \(A\).

1) What is the greatest element \& least element in the Poset \((\mathbb{Z}, \leq)\)?

\[ \text{Poset} \ (\mathbb{Z}, \leq) \rightarrow \text{least element} : 1 \ (C: \ 1/0 \ + \ \text{next}) \]

\[ \text{greatest element: No } C: \text{ there is no int that is } \text{dwell} \text{ by A +ve integers.} \]

2) What is the greatest element \& least element in the Poset \((\mathbb{P}, \subseteq)\)?

\[ \text{Poset} \ (\mathbb{P}, \subseteq) \rightarrow \text{least element: Empty set, } \emptyset \ (C: \text{Empty set is } \subseteq \text{ of all sets}) \]

\[ \text{greatest element: Un set A } \subseteq \ (C: A \text{ is the superset of all possible subsets.}) \]

3) What is least element \& greatest element for the following Hasse Diagrams?

- (a)
  - Least element: 'a'
  - Greatest element: 'd'

- (b)
  - Least element: 'a'
  - Greatest element: 'c'

- (c)
  - Least element: 'a'
  - Greatest element: 'd'

- (d)
  - Least element: 'c'
  - Greatest element: 'd'

- (e)
  - Least element: 'a'
  - Greatest element: 'c'

- (f)
  - Least element: 'a'
  - Greatest element: 'c'

- (g)
  - Least element: 'a'
  - Greatest element: 'c'

- (h)
  - Least element: 'a'
  - Greatest element: 'c'

- (i)
  - Least element: 'a'
  - Greatest element: 'c'

- (j)
  - Least element: 'a'
  - Greatest element: 'c'

- (k)
  - Least element: 'a'
  - Greatest element: 'c'

- (l)
  - Least element: 'a'
  - Greatest element: 'c'

- (m)
  - Least element: 'a'
  - Greatest element: 'c'

- (n)
  - Least element: 'a'
  - Greatest element: 'c'

- (o)
  - Least element: 'a'
  - Greatest element: 'c'

- (p)
  - Least element: 'a'
  - Greatest element: 'c'

- (q)
  - Least element: 'a'
  - Greatest element: 'c'

- (r)
  - Least element: 'a'
  - Greatest element: 'c'

- (s)
  - Least element: 'a'
  - Greatest element: 'c'

- (t)
  - Least element: 'a'
  - Greatest element: 'c'

- (u)
  - Least element: 'a'
  - Greatest element: 'c'

- (v)
  - Least element: 'a'
  - Greatest element: 'c'

- (w)
  - Least element: 'a'
  - Greatest element: 'c'

- (x)
  - Least element: 'a'
  - Greatest element: 'c'

- (y)
  - Least element: 'a'
  - Greatest element: 'c'

- (z)
  - Least element: 'a'
  - Greatest element: 'c'
4. Which elements of the poset \( \{3, 4, 5, 10, 12, 20, 25, 31\} \) are maximal & which are minimal?

So! The elements 3, 5, 12, 20, and 25 are maximal elements.

**Maximal elements:** 12, 20, 25

**Minimal element:** 2 and 5.

Note: A poset can have more than one maximal element and more than one minimal element.

5. Draw the Hasse diagram representing the partial ordering \( \{(a, b) : a \leq b\} \) on the power set \( 2^S \) where \( S = \{a, b, c\} \). Find the maximal, minimal, greatest & least elements of the poset.

So! \( S = \{a, b, c\} \)

**Maximal elements:** \( \{a, b, c\} \)

**Minimal element:** \( \emptyset \)

Here, least element \( = \emptyset = 0 \) element.

w. Greatest element \( = S = \{a, b, c\} = 1 \) element.

w. Minimal element \( = \emptyset \).

w. Maximal element \( = S \).

6. Determine whether the posets represented by each of the Hasse diagrams have a greatest, least, minimal, & maximal element.

- **Diagram 1:** (a) Least Element: a; Greatest Element: No; Minimal Element: a; Maximal Element: b, c, d.
- **Diagram 2:** (a) Least Element: No; Greatest Element: No; Minimal Element: a, b; Maximal Element: d, e.
- **Diagram 3:** (a) Least Element: No; Greatest Element: No; Minimal Element: a, b; Maximal Element: d, e.
- **Diagram 4:** (a) Least Element: a; Greatest Element: d; Minimal Element: a; Maximal Element: d.

7. Find the GLB & LUB.

- **Set A:** \{1, 2, 3, 4, 5\} & B = \{3, 4, 5\}
- **Set B:** \{1, 2, 3\} & C = \{4, 5\}
- **Set C:** \{2, 3\} & D = \{1, 2\}

**GLB(A, B) = 1, 2, 3**

**GLB(B, C) = 3**

**GLB(C, D) = NIL**

**LUB(A, B) = 3, 4, 5**
A lattice is a partially ordered set (poset) \((L, \leq)\) in which for every pair of elements \(a, b \in L\) both greatest lower bound (GLB) \(a \wedge b\) and least upper bound (LUB) \(a \vee b\) exist.

**NOTE:**

1. \(\text{GLB} \{a, b\} = a \wedge b \forall a \neq b \Rightarrow a \text{ meet } b \quad \text{Product } b\)
2. \(\text{LUB} \{a, b\} = a \vee b \forall a \neq b \Rightarrow a \text{ join } b \quad \text{Sum } b\)
3. Lattice is denoted by triplet: \((L, \wedge, \vee)\) or \((L, \wedge, V)\) (\(\wedge, \vee\) are \(\leq\) and \(\geq\) relations).

**Example:** (\(P(\mathbb{C})\), \(\subseteq\)) is a lattice.

- \(\text{LUB} \{a, b\} = a \vee b \quad \text{"LUB} \neq \text{GLB}\) exist.
- \(\text{GLB} \{a, b\} = a \wedge b\)

**Example 2:** The poset \((\mathbb{Z}^+, \mid)\) is a lattice, where \(\mathbb{Z}^+\) denote these.

LUB \(\{a, b\} = \text{L.C.M.}\) of \(a, b\) \(\forall a, b \in \mathbb{Z}^+, \text{gcd}(a, b) = 1\).

**Example 3:** The poset \((\mathbb{Z}, \leq)\) is a lattice, let \(n\) be a positive integer. Let \(D_n\) be the set of all the divisors of \(n\). Then \(D_n\) is a lattice under the relation of divisibility.

**Example 4:** Determine whether the following Hasse diagram of poset lattice.

(a) \(\text{LUB} \{e, f\} = \text{Does not exist}\)
(b) \(\text{LUB} \{24, 36\} = \text{Does not exist}\)
(c) \(\text{LUB} \{a, b\} = \text{Does not exist}\)

The figure shows a lattice.

**Properties of Lattice \((L, \wedge, \vee)\)**

Let \((L, \wedge, \vee)\) be a given lattice. Then \(\wedge\) \& \(\vee\) satisfies the following conditions: \(\forall a, b, c \in L\)

- **Idempotent Law:** \(a \wedge a = a\)
- **Commutative Law:** \(a \wedge b = b \wedge a\)
- **Associative Law:** \((a \wedge b) \wedge c = a \wedge (b \wedge c)\)
- **Absorption Law:** \(a \wedge (a \vee b) = a\), \(a \vee (a \wedge b) = a\)
**Theorem 2.** Let $(L, \wedge, \vee)$ be a lattice in which $\wedge$ and $\vee$ denote the operations meet and join respectively. For any $a, b \in L$,

$$a \leq b \iff a \wedge b = b \iff a \vee b = a$$  \hspace{1cm} (1)

In other words $a \leq b \iff a \vee b = b$.

(b) $a \wedge b = a \iff a \leq b$ \hspace{1cm} (2)

(c) $a \vee b = a \iff a \leq b$ \hspace{1cm} (3)

**Proof.**

(i) $P \rightarrow (a) \rightarrow (b)$

Let $a \leq b$. Since $a$ is the LUB of $\{a, b\}$, $a \leq b$.

Since $a \wedge b$ is the LUB of $\{a, b\}$,

$\Rightarrow a \leq a \wedge b$ \hspace{1cm} (4)

From (1) and (2), we get $a \wedge b = b \Rightarrow a \leq b$. \hspace{1cm} (5)

Let $a \wedge b = b \Rightarrow a \leq b$. \hspace{1cm} (6)

Then $\exists \text{ lower bound of } \{a, b\} = a$.

$\Rightarrow a \leq b$.

**Theorem 3.** Every finite lattice is bounded.

**Proof:**

Let $(L, \wedge, \vee)$ be a given lattice.

Since $L$ is a lattice both $\text{GLB}$ and $\text{LUB}$ exist.

Let "a" be $\text{GLB}$ of $a$ and "b" be $\text{LUB}$ of $a$.

Then for any $x \in L$, we have $a \leq x \leq b$ \hspace{1cm} (7)

From (7):

$\text{GLB} \{a, x\} = a \wedge x = a$ \hspace{1cm} (8)

$\text{LUB} \{a, x\} = a \vee x = a$ \hspace{1cm} (9)

value: Any finite lattice is bounded.

**Theorem 4.** State and prove isotonicity property of lattice.

**Standing:**

Let $(L, \wedge, \vee)$ be given lattice. For any $a, b, c \in L$, we have:

$\text{If } b \leq c \Rightarrow$

$\text{1. } a \wedge b \leq a \wedge c$ \hspace{1cm} (10)

$\text{2. } a \vee b \leq a \vee c$ \hspace{1cm} (11)

$L \text{ and } V \text{ have } a \vee b \leq a \vee c \Rightarrow a \wedge b \leq a \wedge c$.

$a \vee b \leq a \wedge c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
\[ \text{PROOF:} \]

Given \( b \leq c \) \( \Rightarrow \) \( a \land b \leq a \land c \)

\[ \text{let } \begin{align*}
\text{T.P. :} & \quad a \land b \leq a \land c \\
\text{L.H.S. :} & \quad (a \land b) \lor (a \land c) = a \land b \\
\text{R.H.S. :} & \quad a \land (c \land b) = a \land b
\end{align*} \]

\[ \text{using (\( \land \))} \]

\[ \begin{align*}
\text{T.P. :} & \quad a \land b \leq a \land c \\
\text{L.H.S. :} & \quad (a \land b) \lor (a \land c) = a \land b \\
\text{R.H.S. :} & \quad a \land (c \land b) = a \land b
\end{align*} \]

\[ \text{THEOREM 4, SM:} \]

State and Prove distributive Inequality of Lattice.

\[ \text{STATEMENT:} \quad 2\text{MO} \]

Let \( \mathcal{L} \) be a given lattice. For any \( a, b, c \in \mathcal{L} \), the following inequality holds:

\[ a \lor (b \land c) \leq (a \lor b) \land (a \lor c) \]
\[ a \land (b \lor c) \geq (a \land b) \lor (a \land c) \]

\[ \text{PROOF:} \quad \begin{align*}
\text{T.P. :} & \quad a \lor (b \land c) \leq (a \lor b) \land (a \lor c) \\
\text{From the definition of LUB, it is obvious that:} & \quad a \leq a \lor b \Rightarrow 0 \text{ and } b \leq c \Rightarrow b \leq a \lor b \Rightarrow b \leq a \lor c
\end{align*} \]

\[ \text{Hence} \quad a \lor b \leq a \lor c \lor b \land c \Rightarrow \]
From the definition of \( \text{LUB} \), it is obvious that:

\[ \text{LUB}(a, b) \leq c \quad \text{and} \quad \text{LUB}(b, c) \leq c \leq \text{LUB}(a, c) \quad \rightarrow (4) \]

From (3) and (4), \( \text{LUB}(a, b) \leq c \leq \text{LUB}(a, c) \).

Hence, \( \text{LUB}(a, b) \leq \text{LUB}(a, c) \) \( \rightarrow (5) \).

From (4) and (5), \( \text{LUB}(b, c) \leq \text{LUB}(a, c) \).

Hence, \( \text{LUB}(b, c) \leq \text{LUB}(a, c) \).

The duality principle states that:

\[ a \land (b \lor c) = (a \land b) \lor (a \land c) \]

When \( \leq \) is a partial order relation on a set \( S \),

then \( \geq \) is also a partial order relation on \( S \).

To obtain the dual of a lattice \((L, \leq)\), replace:

\[ a \land b \lor c \leq (a \lor b) \land c \]

For example, dual of \( a \leq b \) is \( a \geq b \).

**Theorem:**

Prove that any chain is a distributive lattice.

**Proof:**

Let \((L, \land, \lor)\) be a given chain and \( a, b, c \in L \).

Define chain if any 2 elements are comparable.

Let \( a \leq b \) and \( c \leq d \).

Case I: \( a \leq b \)

Then \( \text{LUB}(a, b) = a \)

\( \text{GLB}(a, b) = a \)

In both cases, any 2 elements of a chain have both \( \text{LUB} \) and \( \text{GLB} \).

Any chain is a lattice. Hence, every chain is a lattice.

\[ L, \lor, \land \] satisfies distributive property.

Let \( a, b, c \in L \).

Since any chain satisfies comparability, we have 6 cases.

Case 1: \( a \leq b \land c \)

Case 2: \( a \leq b \leq c \)

Case 3: \( b \leq a \land c \)

Case 4: \( b \leq a \land c \)

Case 5: \( a \leq b \land c \)

Case 6: \( a \leq b \land c \)
Theorem 5: Modular Inequality

\( (\text{LIAV}) \text{ is a lattice, then for any } a, b, c \in L, \)
\[ a \leq c \Rightarrow a \lor (b \land c) \leq (a \lor b) \land c \]

Proof: Assume \( a \leq c \). T.P. \( a \lor (b \land c) \leq (a \lor b) \land c \)
\[ a \leq c \Rightarrow a \lor c = c \Rightarrow 0 \]
By distributive inequality, we have
\[ a \lor (b \land c) \leq (a \lor b) \land c \]
\[ \Rightarrow (a \leq c) \Rightarrow a \lor (b \land c) \leq (a \lor b) \land c \]
Conversely, assume \( a \lor (b \land c) \leq (a \lor b) \land c \). T.P. \( a \leq c \)
\[ a \lor (b \land c) \leq (a \lor b) \land c \Rightarrow a \leq a \lor (b \land c) \leq (a \lor b) \land c \Rightarrow c \]
\[ \Rightarrow a \leq c \]
From (2) and (3) we have
\[ a \leq c \Rightarrow a \lor (b \land c) \leq (a \lor b) \land c \]

Theorem 6: In a lattice if \( a \leq b \leq c \), then \( T.a \) \( a \lor b = b \lor c \)

Proof:

1. \( a \leq b \leq c \), T.P. \( a \lor b = b \lor c \)
   \[ a \leq b \Rightarrow a \lor b = b \Rightarrow 0 \]
   \[ b \leq c \Rightarrow b \lor c = c \Rightarrow 0 \]
   \[ a \leq c \Rightarrow a \lor c = c \Rightarrow 0 \]
   From (1)-(3), \( a \lor b = b \lor c \)

2. \( a \leq b \leq c \), T.P. \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \)
   \[ L.H.S = (a \lor b) \land (a \lor c) \Rightarrow b \land c \Rightarrow 0 \]
   \[ R.H.S = (a \lor b) \land (a \lor c) \Rightarrow b \land c = b \]
   From (4) and (5), \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \)
**Xen. Proof for Properties of Lattice**

Let \((L, \leq)\) be a lattice. If \(a\leq b\) define \(a \land b = \text{glb}(a,b)\) and \(a \lor b = \text{lub}(a,b)\). Then show that both \(\land\) and \(\lor\) satisfy commutative, associative, absorption, and idempotent laws.

**Property 1: Idempotent Law**

Let \((L, \leq)\) be a given lattice. Then for any \(a,b,c \in L\),
\[
\begin{align*}
    a \land a &= a \quad \text{and} \quad a \lor a &= a \\
    a \land (a \lor b) &= a \\
    a \lor (a \land b) &= a
\end{align*}
\]

**Proof:**

\[
\begin{align*}
    a \land a &= \text{glb}(a,a) = \text{glb}(a) = a \\
    a \lor a &= \text{lub}(a,a) = \text{lub}(a) = a \\
    a \land (a \lor b) &= \text{glb}(a, a \lor b) = \text{glb}(a) = a \\
    a \lor (a \land b) &= \text{lub}(a, a \land b) = \text{lub}(a) = a
\end{align*}
\]

**Property 2: Commutative Law**

Let \((L, \leq)\) be a given lattice. Then for any \(a,b,c \in L\),
\[
\begin{align*}
    a \land b &= b \land a \\
    a \lor b &= b \lor a
\end{align*}
\]

**Proof:**

\[
\begin{align*}
    a \land b &= \text{glb}(a,b) = \text{glb}(b,a) = b \land a \\
    a \lor b &= \text{lub}(a,b) = \text{lub}(b,a) = b \lor a
\end{align*}
\]

**Property 3: Associative Law**

Let \((L, \leq)\) be a given lattice. Then for any \(a,b,c \in L\),
\[
\begin{align*}
    a \land (b \land c) &= (a \land b) \land c \\
    a \lor (b \lor c) &= (a \lor b) \lor c
\end{align*}
\]

**Proof:**

\[
\begin{align*}
    a \land (b \land c) &= a \land (\text{glb}(b,c)) = \text{glb}(a,b,c) = (a \land b) \land c \\
    a \lor (b \lor c) &= a \lor (\text{lub}(b,c)) = \text{lub}(a,b,c) = (a \lor b) \lor c
\end{align*}
\]

**Property 4: Absorption Law**

Let \((L, \leq)\) be a given lattice. Then for any \(a,b,c \in L\),
\[
\begin{align*}
    a \land (a \lor b) &= a \\
    a \lor (a \land b) &= a
\end{align*}
\]

**Proof:**

\[
\begin{align*}
    a \land (a \lor b) &= a \land \text{lub}(a,b) = \text{glb}(a,b) = a \\
    a \lor (a \land b) &= a \lor \text{glb}(a,b) = \text{lub}(a,b) = a
\end{align*}
\]

**Def: Distributive Lattice**

A lattice \((L, \leq)\) is said to be distributive if \(a \land (b \lor c) = (a \land b) \lor (a \land c)\) and \(a \lor (b \land c) = (a \lor b) \land (a \lor c)\).

\[
\begin{align*}
    a \land (b \lor c) &= (a \land b) \lor (a \land c) \\
    a \lor (b \land c) &= (a \lor b) \land (a \lor c)
\end{align*}
\]

**Note:** If either \(D_1\) or \(D_2\) holds, the lattice is distributive.

**Ref:**

- If \(a \leq c\), then \(a \land (b \lor c) = (a \land b) \lor (a \land c)\)

**Def: Modular Lattice**

A lattice \((L, \leq)\) is said to be modular if it satisfies the following condition:

\[a \land (b \lor c) = (a \land b) \lor (a \land c)\]

**Note:**

- If \(a \leq c\), then \(a \land (b \lor c) = (a \land b) \lor (a \land c)\)
1. Check the pentagon lattice. It is modular or distributive.

Consider \((a_1, a_2, a_3)\):

- L.H.S. \(a_1 \lor (a_2 \land a_3) = a_1 \lor a_3 = a_1\)
- R.H.S. \((a_1 \lor a_2) \land a_3 = a_2 \land a_3 = a_3\)

Hence, the lattice is not distributive.

MODULAR PROPERTY: \(M_1\) : If \(a \leq c \Rightarrow a \lor (b \land c) = (a \lor b) \land c\)

L.H.S. \(a \lor (b \land c) = a \lor c\) \(\Rightarrow 0\) \(\text{if } a \neq c\)

R.H.S. \((a \lor b) \land c = c\)

Hence, the lattice is not modular.

Theorem B & M: Every distributive lattice is modular, but not conversely.

Proof:

Let \((L, \lor, \land)\) be the given distributive lattice.

\(D_1: a \lor (b \land c) = (a \lor b) \land (a \lor c)\) \(\Rightarrow \) \(a, b, c \in L\)

Now, if \(a \leq c\), then \(a \lor c = c\) \(\Rightarrow 2\)

\(D_2: a \lor (b \land c) = (a \lor b) \land (a \lor c)\) \(\Rightarrow 2\)

If \(a \leq c\), then \(a \lor (b \land c) = (a \lor b) \land c\).

Every distributive lattice is modular.

But the converse is not true. Every modular lattice need not be distributive.

Example: M5 or Diamond lattice is an example of a modular lattice but non-distributive lattice.

Theorem: Cancellation Law: In any distributive lattice \((L, \lor, \land)\), for \(a, b, c \in L\), prove that if 

\(b = a \lor c\) then \(a \land b = a \land c \Rightarrow b = c\)

Proof:

Consider \(b = b \lor (b \land c)\) (Absorption Law)

- \(= b \lor (a \land b)\) (Commutative Law)
- \(= (b \lor a) \land (b \lor b)\) (Distributive Law)
- \(= (a \lor b) \land (b \lor c)\) (Commutative Law)
- \(= (a \lor c) \land (b \lor c)\) (Commutative Law)

\(\text{Hence, } a \land b = a \land c \Rightarrow b = c\)
\[ \begin{align*} 
&= (c \lor a) \land (c \lor b) \quad \text{(Commutative Rule)} \\
&= c \land (c \lor a) \lor b \quad \text{(Distributive, Cond.: D)} \\
&= c \lor (c \land a) \quad \text{(Comm. law, Cond.)} \\
&= c \lor (c \land a) \\
&= c \\
\end{align*} \]

\[ b = c \]

**Problem**

Show that in a lattice if \( a \leq b \) and \( c \leq d \), then \( a \land c \leq b \lor d \).

**Solution**

If \( a \leq b \) and \( c \leq d \), then \( a \lor c \leq b \lor d \).

L.H.S. = \( (a \land c) \lor (b \lor d) \)

\[ \begin{align*} 
&= (a \land c) \lor (b \lor d) \\
&= a \land c \lor b \lor d \\
&= (a \lor b) \land (c \lor d) \\
&= a \land c \\
&= \text{By Def.} \\
&= a \land c \leq b \lor d. \\
\end{align*} \]

**Theorem:** If in any distributive lattice \((L, \lor, \land)\), prove that \( (a \land b) \lor (b \lor c) \land (c \lor a) = (a \lor b) \lor (b \lor c) \land (c \lor a) \).

**Proof**

L.H.S. = \( (a \land b) \lor (b \lor c) \land (c \lor a) \)

\[ \begin{align*} 
&= (a \land b) \lor (b \lor c) \land (c \lor a) \\
&= (a \land b) \lor (b \lor c) \land (c \lor a) \\
&= (a \land b) \lor (b \lor c) \land (c \lor a) \\
&= (a \land b) \lor (b \lor c) \land (c \lor a) \\
&= \text{R.H.S.} \\
&= (a \land b) \lor (b \lor c) \land (c \lor a) = (a \lor b) \lor (b \lor c) \land (c \lor a). \\
\end{align*} \]

**Lattice as a Algebraic System**

**Definition:** A lattice is an algebraic system \((L, \lor, \land)\) with two binary operations \( \lor \) and \( \land \) on \( L \) which are both commutative, associative, and satisfy absorption laws. An \( S \) is a lattice if it is \( D \)-closed under \( \lor \) and \( \land \) operations.

**Sublattice**

If \((L, \lor, \land)\) be a lattice and let \( S \subseteq L \) be a subset of \( L \). Then \((S, \lor, \land)\) is a sublattice of \((L, \lor, \land)\) if \( S \) is closed under \( \lor \) and \( \land \) operations.
DEF: LATTICE HOMOMOPHISM
Let \((L_1, \vee, \wedge, \cdot, 0, 1)\) and \((L_2, \vee, \wedge, \cdot, 0, 1)\) be two given lattices.
A mapping \(f: L_1 \to L_2\) is called lattice homomorphism if, for all \(a, b \in L_1\),
1. \(f(a \vee b) = f(a) \vee f(b)\)
2. \(f(a \wedge b) = f(a) \wedge f(b)\)

DEF: LATTICE ISOMORPHISM
A homomorphism which is also 1-1 is called lattice isomorphism.

DEF: DIRECT PRODUCT OF LATTICE
Let \((L_1, \vee, \wedge)\) and \((L_2, \vee, \wedge)\) be two lattices. The algebraic system \((L_1 \times L_2, \cdot, +)\) in which the binary operation \(+\) on \(L_1 \times L_2\) are such that for any \((a_1, b_1)\) and \((a_2, b_2)\) in \(L_1 \times L_2\)
\[(a_1, b_1) + (a_2, b_2) = (a_1 \vee a_2, b_1 \wedge b_2)\]
\[(a_1, b_1) \cdot (a_2, b_2) = (a_1 \wedge a_2, b_1 \vee b_2)\]
is called the direct product of the lattice \((L_1, \vee, \wedge)\) and \((L_2, \vee, \wedge)\).

THEOREM: The direct product of any two distributive lattices is also a lattice.

If \((L_1, \vee, \wedge)\) and \((L_2, \vee, \wedge)\) are two distributive lattices, then \((L_1 \times L_2, \vee, \wedge)\) is also a lattice.

PROOF:
1. \(\cdot\) and \(\wedge\) satisfies commutative, associative and absorption in \(L_1 \times L_2\)
2. \((x_1, y_1), (x_2, y_2) \in L_1 \times L_2\)
   \[x_1 + y_1, x_2 + y_2 \text{ are commutative}\]
   \[x_1 (x_2 + y_2) = (x_1 \cdot x_2) + y_1 (y_2)\]
   \[x_1 y_1 + x_1 y_2 = (x_1 y_1) + (x_1 y_2)\]
   \[x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2 \text{ are associative}\]
   \[x_1 y_1 + [x_1 y_2 + x_2 y_3] = (x_1 y_1) + x_2 (x_2 y_3)\]
   \[x_2 y_1 + (x_2 y_2) + x_3 y_3\]
   \[x_1 y_1 + x_2 y_2 + x_3 y_3\]
$$\text{and satisfies absorption law}$$

$$\begin{align*}
    (x, y_1) + [(x, y_1) \cdot (x, y_2)] &= (x, y_1) + [x_1, y_2] \\
    &= (x, y_1, y_2) \\
    (x, y_1) + [(x, y_1) \cdot (x, y_2)] &= (x, y_1) \\
    \text{and satisfies absorption law.}
\end{align*}$$

Hence $(L, \leq, *)$ is a lattice.

**SOME SPECIAL LATTICE**

DEF: **Bounded Lattice** $(L, \leq, 1, 0)$

Let $(L, \leq)$ be a given lattice. If it has both $1$ and $0$ element then it is said to be a bounded lattice.

Eq: $(p(a), \leq)$ is a bounded lattice.

DEF: **Complement of an Element**

Let $(L, \leq, 1, 0)$ be a given bounded lattices. Let $a \in L$.

The element $a'$ is called complement of $a$ if

$$a \land a' = 0 \quad \text{and} \quad a \lor a' = 1$$

DEF: **Complemented Lattice**

A bounded lattice $(L, \leq, 1, 0)$ is said to be a complemented lattice if every element of $L$ has at least one complement.

Eq: check the given lattice is complemented lattice or not.

- For $c \land b$
  - $b \land c = a \land b = 1$
  - $b \land a = a \land b = b$

The given is not complemented lattice.

**THEOREM II: De Morgan's Law of Lattice**

If $(L, \leq, 1, 0)$ is a complemented lattice,

1. $(a \land b)' = a' \lor b'$ or $(a \land b) = a' \lor b'$
2. $(a \lor b)' = a' \land b'$ or $(a \lor b) = a' \land b'$

**Proof**

1. T-P
   - $a \land b = a' \lor b'$

2. T-R
   - $(a \land b) \lor (a \lor b) = 0$
   - $(a \land b) \lor (a \lor b) = 1$

**ALTERNATE STATEMENT OF THEOREM II:**
Shake & Prove De Morgan's law in a bounded, complemented & distributive lattice.
Theorem 2: In a complemented distributive lattice, each element has one complement.

Proof: Let \( x \in L \) be an element of the lattice, and assume \( x \) has two complements, \( y \) and \( z \). Then

1. \( x \lor y = 1 \)
2. \( x \lor z = 1 \)
3. \( y \land z = 0 \)

But by the distributive property,

\[
(x \lor y) \land (x \lor z) = x \lor (y \land z) = x \lor 0 = x.
\]

This contradicts the assumption that \( x \lor y = 1 \) and \( x \lor z = 1 \), unless \( y = z \) is the unique complement of \( x \).
\[ y = y \lor 0 = y \lor (y \land x) \quad \text{[using 2]} \\
   = y \lor (y \land x) \quad \text{[Dist law]} \\
   = (y \lor y) \land (y \lor x) \quad \text{[Commutative law]} \\
   = 1 \land (y \lor x) \quad \text{[using (2)]} \\
   y = y \lor x \quad \text{- (13)} \\
\]

From (5) & (8) we have \( x = x \lor y = y \lor x = y \lor y = y \) 
\[ x = y \]

Complement is unique.

\textbf{Problem}

Show that in a distributive and complemented lattice \( a \leq b \Rightarrow a \lor b' = 0 \Leftrightarrow a \land b = 1 \Rightarrow b = a' \) (14)

\( a \leq b \Rightarrow a \land b = 0 \Leftrightarrow a \lor b = 1 \Rightarrow b \leq a' \) (i) (ii) (iii) (iv) (v)

\textbf{Scl}

1. \textbf{T.P. (i) \Rightarrow (ii)} \quad \text{Let } a \leq b \Rightarrow a \land b = 0

\[ a \land b = (a \lor b) \land (a \lor b') \quad \text{[using (1)]} \\
   = (a \lor b) \land a' \quad \text{[Ass Law]} \\
   = (a \land a') \land (a \lor b) \\
   = 0 \land (a \lor b) \\
   = 0 \\
   a \leq b \Rightarrow a \land b = 0 \]

2. \textbf{T.P. (ii) \Rightarrow (iii)} \quad \text{Let } a \land b = 0 \Rightarrow a' \lor b = 1

\[ a' \lor b = 1 \\
   \Downarrow a \land b = a \land b' = 0 \\
\]

3. \textbf{T.P. (iii) \Rightarrow (iv)} \quad \text{Let } a' \lor b = 1 \Rightarrow b \leq a'

\[ a' \lor b = 1 \Rightarrow (a \lor b) \land b' = 1 \land b' \quad \text{[By cancellation law]} \\
   = (a \lor b) \land (a' \lor b') = b' \quad \text{[Dist law]} \\
   = c(a \land b') \lor b = b' \quad \text{[Dist law]} \\
   = a' \lor b = b' \\
   b \leq a' \Rightarrow a' \lor b = b' \]

4. \textbf{T.P. (iv) \Rightarrow (v)} \quad \text{Let } b \leq a' \Rightarrow a \leq b

\[ b \leq a' \Rightarrow a \lor b = b' \quad \text{[Ass Law]} \\
\]

5. \textbf{T.P. (v) \Rightarrow (vi)} \quad \text{Let } a \leq b \Rightarrow a \lor b \leq b \Rightarrow a \lor b = b, a \lor b = b \Rightarrow a \leq b \]
3. If $S_{42}$ is the set of all divisors of 42 and $D$ is the relation "divisor of" on $S_{42}$, prove that $S_{42}, D$ is a complemented lattice.

Sol: $S_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$.

The Hasse diagram for $(S_{42}, D)$ is:

Here, 1 is the greatest element and 42 is the least element.

- $\text{lcm}(3, 14) = \gcd(3, 14) = 1$
- $\text{lcm}(7, 14) = \gcd(7, 14) = 1$
- $\text{lcm}(14, 21) = \gcd(14, 21) = 1$
- $\text{lcm}(2, 14) = \gcd(2, 14) = 1$
- $\text{lcm}(14, 42) = \gcd(14, 42) = 1$
- $\text{lcm}(2, 42) = \gcd(2, 42) = 1$

By definition, $(S_{42}, D)$ is a complemented lattice.

4. If $S_n$ is the set of all divisors of the positive integer $n$, $D$ is the relation "divisor of". Find the sublattices of $(S_{30}, D)$ that contain 6 or more elements.

Sol: Let $a, b \in S_{30}$.

$\text{lcm}(a, b) = \gcd(a, b)$.

Sublattice $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

- Sublattice with 6 elements:
  $S_2 = \{1, 2, 3, 6\}$
  $S_3 = \{1, 3, 6\}$
  $S_6 = \{1, 2, 3, 6\}$

$S_2 \times S_3 \times S_6$ is called a sublattice of $S_{30}$, but $1$ is odd.

$S_{15} = \{1, 2, 5, 10\}$

$S_{15} \times S_{30}$ is not a sublattice.

$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Sublattice with 6 more elements:

$S_{30} = \{1, 2, 3, 10, 15, 30\}$

$S_{30} \times S_{30}$ is not a sublattice.
A Complemented distributive Lattice is called Boolean Algebra (or)

A Boolean Algebra is a non-empty set with 2 binary operations \( \land \) and \( \lor \) and is satisfied by the following conditions:

1. \( a \land a = a \)
2. \( a \lor b = b \lor a \)
3. \( a \land (b \lor c) = (a \land b) \lor (a \land c) \)
4. \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \)
5. \( a \land 0 = 0 \)
6. \( a \lor 1 = 1 \)

Theorem 1: Idempotent Law: \( a \land a = a \) and \( a \lor a = a \)

Proof: \( a \land a = a \)

By Identity axiom, \( a = a + 0 \)

\( a = a \land a \lor a \) \([\text{Complement Law]}\]

\( = (a \land a) \lor (a \land a) \) \([\text{Distributive Law]}\]

\( = (a \land a) \lor (a \land a) \) \([\text{Complement Law]}\]

\( = a \land a \) \([\text{Complement Law]}\]

\( = a \land a \lor a \) \([\text{Complement Law]}\]

\( = a \land a \lor a \) \([\text{Complement Law]}\]

Theorem 2: The set \( 0 \) of a Boolean Algebra \( B \) are unique.

Proof: \( \forall e \in B \) \( e \lor 0 = 1 \) \([\text{Identity Law]}\)

Assume \( 0, 0' \) be two zero element in \( B \)

\( 0 \lor 0 = 1 \)

\( 0' \lor 0 = 1 \)

Commutative law \( 0 \lor 0' = 0 \lor 0 \)

\( \Rightarrow 0 = 0' \)

\( 0' \) is unique

Theorem 3: In Boolean Algebra \( B \), \( 0' = 1 \) \( \text{and} \ 1' = 0 \)

Proof:

We have \( 0' = 0 + 0 \) \([\text{Identity Law]}\)

\( 0' = 0 \) \([\text{Complement Law]}\)

\( 1' = 1 \) \([\text{Identity Law]}\)

\( 1' = 0 \) \([\text{Complement Law]}\)
**Theorem 4: Boundedness Laws (or Dominant Law)**

(i) \( a + 1 = 1 \) and (ii) \( a \cdot 0 = 0 \neq a \cdot 1 \).

**Proof**

(i) T.P. \( a + 1 = 1 \) \( \forall a \in B \)

\[
\begin{align*}
\text{Proof:} & \\
\text{Let } a + 1 & = 1 \\
\text{Then, } a + (a + 1) & = (a + 1) + a \\
& = a + a' + 1 \\
& = a + (a + a') \\
& = a + 1 \\
& = 1 \\
\end{align*}
\]

(ii) T.P. \( a \cdot 0 = 0 \) \( \forall a \in B \)

\[
\begin{align*}
\text{Proof:} & \\
\text{Let } a \cdot 0 & = 0 \\
\text{Then, } a \cdot (a \cdot 0) & = (a \cdot 0) \cdot a \\
& = a \cdot 1 \\
& = a \\
\end{align*}
\]

**Theorem 5: Unique Complement**

In a Boolean algebra, \( B \), complement of every element is unique.

**Proof**

Let \( a \in B \) be any element. If \( a_1 \) and \( a_2 \) be two complements of \( a \), then

\[
\begin{align*}
\text{Let } a + a_1 & = 1 \\
\text{and } a \cdot a_1 & = 0 \\
\text{Then, } a + a_2 & = 1 \\
\text{and } a \cdot a_2 & = 0 \\
\end{align*}
\]

Now \( a_2 = (a_2 \cdot a_1) \).

\[
\begin{align*}
\text{Identity Law} & \\
& = a_2 \cdot a + a_2 \cdot a_1 \\
& = a_2 \cdot a + a_1 \\
& = a \cdot a_2 + a_1 \\
& = a + a_1 \\
& = a \cdot a_2 + a_1 \\
& = a \cdot a_2 + a_1 \\
& = a_2 \\
\end{align*}
\]

Thus, complement of an element is unique.

**Theorem 6: Absorption Law**

(i) \( a \cdot (a + b) = a \) \( \forall a \in B \)

(ii) \( a + (a \cdot b) = a \cdot a + b \in B \)

**Proof**

(i) T.P. \( a \cdot (a + b) = a \)

\[
\begin{align*}
\text{Proof:} & \\
\text{Let } a \cdot (a + b) & = a \\
\text{Then, } (a \cdot a) + (a \cdot b) & = a \\
& = a + b \\
& = a \\
\end{align*}
\]

(ii) T.P. \( a + a \cdot b = a \)

\[
\begin{align*}
\text{Proof:} & \\
\text{Let } a + a \cdot b & = a \\
\text{Then, } (a + a) \cdot (a + b) & = a \\
& = a \cdot b \\
& = a \\
\end{align*}
\]
Theorem 7: De Morgan's Law

\[ (a \land b)' = a' \lor b' \]

Proof:

1. \( a \land (b' \lor c') = (a \land b') \lor (a \land c') \)
2. \( (a \land b') = a' \lor b' \) (Complement of \( a \land b \))
3. \( a \lor b = a \lor b' \) (Complement of \( a \lor b \))
4. \( a \land b' = a' \lor b' \) (Complement of \( a \land b' \))
5. \( a' \lor b' = a' \lor b' \) (Complement of \( a \lor b \))

Thus, \( a \land b = a' \lor b' \)

(1) \( T \cdot P \ (a \land b)' = a' \lor b' \) (Complement of \( a \land b \))

(2) \( T \cdot P \ (a \lor b)' = a' \land b' \) (Complement of \( a \lor b \))

Now, \( a \land b + c = a \land b' \)

\[ = [a \land (a' \lor b')] \lor [b + (a' \lor b')] \] (Distributive Law)

\[ = [(a \land a') \lor b'] \lor [b + (a' \lor b')] \] (Associative Law)

\[ = [(a \land a') \lor b'] \lor [b + (a' \lor b')] \] (Commutative Law)

\[ = [1 \lor b'] \lor [1 + a'] \] (Bound Law)

\[ = [b' \lor b] \lor [a'] \] (Idem Law)

Thus, \( a \land b + c = a \land b' \)

\[ (a \land b)' = a' \lor b' \]