UNIT III

Classification of Random Processes.

Stationary process:

Random Variable:

A random variable is a rule that assigns a real number to every outcome of a random experiment.

Random process:

A random process is a rule that assigns a time function to every outcome of a random experiment.

A random process is a collection of random variables \( \{X(t)\} \) where \( t \in \Omega \) (Parameter set).

Classification of Random Processes:

Discrete Random Sequence:

If both \( S \) and \( T \) are discrete then the random process is called discrete random sequence.

Example: No. of books in library at opening time.

Continuous Random Sequence:

If \( S \) is continuous and \( T \) is discrete, then the random process is
Example: Quantity of petrol in the bulk at opening time.

Discrete Random Process:
If \( s \) is discrete and \( t \) is continuous, and the random process is called discrete random process.

Example: No. of phone calls received in (0,t).

Continuous Random Variable Process:
If \( s \) is continuous and \( t \) is continuous, then the random process is called continuous random process.

Example: Stirring sugar in coffee.

Strict Sense Stationary:
A random process is called a strongly stationary process (or strict sense stationary process) if all its finite dimensional distributions are invariant under translation of time parameter.

Note:
- \( \mu_t \) is \( \text{SSS} \)
- \( \mu_t(t) \) is constant
- \( \mu_t(t) \) is constant
- \( \mu_t(t) \) is constant.
Wide Sense Stationary

A random process is called wide sense stationary (WSS) or weakly stationary process (or) covariance stationary process.

(i) \[ E[x(t)] \text{ is constant} \]

(ii) Auto correlation is a function of \( t > t' \) (WSS).

Note:

A random process, non-stationary is called an evolutionary process.

\[ x(t) \text{ and } y(t) \text{ are said to be jointly WSS} \]

(iii) \( R_{xy}(t) \) is a function of \( t \).

(iv) Each process is individually WSS.


1. Show that it is not-stationary. The process \( x(t) \) whose probability distribution under certain conditions is given by

\[
P(x(t) = n) = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2, \ldots \\ \frac{at}{1+at} & n = 0 \\
\frac{1}{1+at} & n < 0 
\end{cases}
\]

Solutions:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(x(t) = n) )</td>
<td>\frac{1}{1+at}</td>
<td>\frac{at}{1+at}</td>
<td>\frac{(at)^{n-1}}{(1+at)^{n+1}}</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
\[ E[x^n(t)] = \sum_{n=0}^{\infty} \frac{t^n}{n!} p(n) \left( \begin{array}{c} \infty \vspace{1cm} \\ n \end{array} \right) \frac{1}{(1+at)^n} \]

\[ = \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) \]

\[ = \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) \]

\[ = \frac{1}{(1+at)^2} \left[ 1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right)^2 + \ldots \right] \]

\[ = \frac{1}{(1+at)^2} \left[ \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right)^2 + \ldots \right] \]

\[ = \frac{1}{(1+at)^2} \left[ \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right)^2 + \ldots \right] \]

\[ = \frac{1}{(1+at)^2} \left[ \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right)^2 + \ldots \right] \]

\[ = \frac{1}{(1+at)^2} \left[ \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left( \frac{at}{1+at} \right)^2 + \ldots \right] \]
Given,

\[ X(t) = \cos(\omega t + \gamma) \]

\[ \Phi(\omega) = E[\cos \omega t + i \sin \omega t] \]

\[ \Phi(0) = 0 \]

\[ \Rightarrow \Phi(1) = E[\cos \gamma + i \sin \gamma] = 0 \]

\[ E[\cos \gamma] + i E[\sin \gamma] = 0 \]

\[ E[\cos \gamma] = 0 \quad \text{and} \quad E[\sin \gamma] = 0 \]

\[ \Phi'(a) = 0 \]

\[ \Rightarrow E[\cos 2\gamma + i \sin 2\gamma] = 0 \]

\[ E[\cos 2\gamma] + i E[\sin 2\gamma] = 0 \]

\[ E[\cos 2\gamma] = 0 \quad \text{and} \quad E[\sin 2\gamma] = 0. \]

(i) \[ E[X(t)] = E[\cos(\omega t + \gamma)] \]

\[ = E[\cos(\omega t) \cos \gamma - \sin(\omega t) \sin \gamma] \]

\[ = \cos \omega t E[\cos \gamma] - \sin \omega t E[\sin \gamma] \]

\[ = 0 \quad \text{constant}, \quad E[\cos \gamma] = 0 \]

\[ E[\sin \gamma] = 0 \]

(ii) \[ R_{xx}(t) = E[X(t) \cdot X(t + \tau)] \]

\[ = E[\cos(\omega t + \gamma) \cdot \cos(\omega (t + \tau) + \gamma)] \]
\[ E \left[ \cos \lambda t \cos \lambda (t + \tau) \cos^2 y + \sin \lambda t \sin \lambda (t + \tau) \sin^2 y + \left( \cos \lambda t \sin \lambda (t + \tau) \cos y \sin y + \sin \lambda t \cos \lambda (t + \tau) \sin y \sin y \right) \right] \\
= E \left[ \cos \lambda t \cos \lambda (t + \tau) \left( \frac{1 + \cos 2y}{2} \right) \right] + E \left[ \sin \lambda t \sin \lambda (t + \tau) \left( \frac{1 - \cos 2y}{2} \right) \right] + E \left[ \sin \lambda t \sin \lambda (t + \tau) \right] \left( \frac{\sin y \sin y}{2} \right) \left( \cos \lambda t \sin \lambda (t + \tau) + \sin \lambda t \cos \lambda (t + \tau) \right) \\
= \frac{1}{2} \left[ \cos \lambda t \cos \lambda (t + \tau) \right] + \frac{1}{2} \left[ \sin \lambda t \sin \lambda (t + \tau) \right] \\
= \frac{1}{2} \left[ \cos \lambda t \cos \lambda (t + \tau) \right] + \frac{1}{2} \left[ \sin \lambda t \sin \lambda (t + \tau) \right] \\
= \frac{1}{2} \cos (\lambda t - \lambda (t + \tau)) + \frac{1}{2} \cos \lambda t, \text{ free from } t \\
\lambda (t) \text{ is WSS.} \]
4. Show that the process \( X(t) = A \cos \lambda t + B \sin \lambda t \)
\((A, B)\) are random variables. It is WSS.

(i) \( E[AJ] = E[BJ] = 0 \)

(ii) \( E[A^2J] = E[B^2J] = \phi \)

(iii) \( E[AB] = 0 \)

(iv) \( E[X(t)] = E[A \cos \lambda t + B \sin \lambda t] \)
\(= \cos \lambda t \cdot E[AJ] + \sin \lambda t \cdot E[BJ] \)
\(= 0, \) constant.

(v) \( R_{XX}(\tau) = E[X(t)] \cdot X(t+\tau) \)
\(= E[A \cos \lambda t + B \sin \lambda t] \cdot [A \cos \lambda (t+\tau) + B \sin \lambda (t+\tau)] \)
\(= E[A^2 \cos \lambda t \cdot \cos \lambda (t+\tau) + B^2 \sin \lambda t \cdot \sin \lambda (t+\tau)] \)
\(+ A B \cos \lambda t \cdot \sin \lambda (t+\tau) + A B \sin \lambda t \cdot \cos \lambda (t+\tau)] \)
\(= \cos \lambda t \cdot \cos \lambda (t+\tau) \cdot E[A^2J] + \sin \lambda t \cdot \sin \lambda (t+\tau) \cdot E[B^2J] + E[AB] \cdot \cos \lambda t \cdot \sin \lambda (t+\tau) + \sin \lambda t \cdot \sin \lambda (t+\tau) \cdot E[AB] \)
\(= \cos \lambda t \cdot \cos \lambda (t+\tau) \cdot \phi + \)
\[ \begin{align*}
\cos \omega t & \cos \alpha (t + \tau) + \sin \omega t \\
\sin \alpha (t + \tau) & = \Phi \left[ \cos (\omega t - (\alpha (t + \tau)) \right] \\
\cos (\omega t - \alpha t - \alpha \tau) & = \Phi \cos \alpha t, \text{ free from } t, \\
\Phi(t) & \text{ is WSS.}
\end{align*} \]

5. Two random processes \( X(t) \) and \( Y(t) \) are given by

\[ X(t) = A \cos \omega t + B \sin \omega t \]
\[ Y(t) = B \cos \omega t - A \sin \omega t \]

Show that, \( X(t) \) and \( Y(t) \) are jointly WSS, i.e., \( X(t) \) and \( Y(t) \) are uncorrelated with zero mean and the same variances with \( \omega \) and \( \alpha \) constant.

Solution:

\[ X(t) = A \cos \omega t + B \sin \omega t \]
\[ Y(t) = B \cos \omega t - A \sin \omega t \]

\[ \text{Var}(X) = \text{Var}(B) = \sigma^2 \]
\[ \Rightarrow E[X^2] = E[Y^2] = \sigma^2 \]
\[ A \text{ and } B \text{ are uncorrelated.} \]
\[ E[X(t)] = E\left[ A \cos \omega t + B \sin \omega t \right] \]
\[ = E[A] \cos \omega t + E[B] \sin \omega t \]
\[ = 0, \text{ constant} \]

\[ R_X(t) = E[X(t) \cdot X(t + \tau)] \]
\[ = E\left[ A \cos \omega t \cdot B \sin \omega t + B \sin \omega t \cdot A \cos \omega t + A \cos \omega t \cdot A \cos \omega (t + \tau) + B \sin \omega t \cdot B \sin \omega (t + \tau) \right] \]
\[ + A B \left( \cos \omega t \sin \omega (t + \tau) + \sin \omega t \cos \omega (t + \tau) \right) \]
\[ = E[A^2] \cos \omega t \cos \omega (t + \tau) + E[B^2] \sin \omega t \sin \omega (t + \tau) \]
\[ + A B \left( \cos \omega t \sin \omega t \cos \omega (t + \tau) + \sin \omega t \cos \omega (t + \tau) \right) \]
\[ = A^2 \left( \cos \omega t \cos \omega (t + \tau) + \sin \omega t \sin \omega (t + \tau) \right) + 0 \]
\[ = \sigma^2 \left( \cos \omega t \cos \omega (t + \tau) + \sin \omega t \sin \omega (t + \tau) \right) \]
\[ = \sigma^2 \cos \omega \tau, \text{ free from } t \]
\[ X(t) \text{ is WSS.} \]
\[ Y(t) = \cos\omega t - A \sin\omega t \]

\[ E[Y(t)] = E[\cos\omega t - A \sin\omega t] \]

\[ = E[\cos\omega t] - E[A \sin\omega t] \]

\[ = 0, \text{ constant} \]

\[ R_{yy}(t) = E[Y(t)Y(t+\tau)] \]

\[ = E[(B \cos(\omega t + A \sin(\omega t)) - A \sin(\omega (t+\tau))] \]

\[ = E[(B^2 \cos(\omega t) \cos(\omega (t+\tau)) + A^2 \sin(\omega t) \sin(\omega (t+\tau))] \]

\[ = E[(B^2 \cos(\omega (t+\tau))] - E[A^2 \sin(\omega (t+\tau))] - E[AB \cos(\omega t) \sin(\omega (t+\tau))] \]

\[ = \sigma^2 \left[ \cos(\omega t + \omega (t+\tau)) + \sin(\omega t + \omega (t+\tau)) \right] \]

\[ = \sigma^2 \left[ cos(\omega (t+\tau)) \right] \]

\[ = \sigma^2 \left[ \cos(\omega (t+\tau)) \right] \]

\[ = \sigma^2 \cos\omega t, \text{ free from } \tau \]
\[ R_{xy}(t) = E \left[ x(t), y(t+\tau) \right] \]

\[ = E \left[ A \cos \omega t + B \sin \omega t \right] \left[ B \cos \omega (t+\tau) - A \sin \omega (t+\tau) \right] \]

\[ = E \left[ B^2 \sin \omega t \cos \omega (t+\tau) - A^2 \cos \omega t \sin \omega (t+\tau) \right] \]

\[ = \sigma^2 \left( \sin \omega (t+\tau) - \cos \omega t \cos \omega (t+\tau) \right) \]

\[ = \sigma^2 \left( \sin \omega (t+\tau) - \cos \omega t \cos \omega (t+\tau) \right) \]

\[ = \sigma^2 \left[ \sin (\omega t - \omega t - \omega t) \right] \]

\[ = \sigma^2 \left[ \sin (-\omega t) \right] \]

\[ = -\sigma^2 \sin \omega t, \text{ free from } \tau. \]

b. If \( x(t) = Y \cos t + Z \sin t + \xi \), where \( Y \) and \( Z \) are independent binary random variables each of which assumes the values \(-1, 1\) with probabilities \(2/3\) and \(1/3\) respectively. Prove that \( x(t) \) is wide.

Solution:

Given

\[ \begin{array}{c|c|c}
Y & -1 & 1 \\
\hline
P(Y) & 1/3 & 2/3 \\
\end{array} \quad \begin{array}{c|c|c}
Z & -1 & 1 \\
\hline
P(Z) & 2/3 & 1/3 \\
\end{array} \]
\[ E \left[ y^2 \right] \cos \theta \cos (x + \theta) + E \left[ y^2 \right] \sin \theta \sin (x + \theta) \\
- E \left[ yz \right] \left[ \cos \theta \sin (x + \theta) + \sin \theta \cos (x + \theta) \right] \\
= 2 \left[ \cos \theta \cos (x + \theta) + \sin \theta \sin (x + \theta) \right] + 0 \\
= 2 \cos (x + \theta) \\
= 2 \cos \theta \\
= 2 \cos \theta. \text{ Free from } \theta. \\
x(t) \text{ is WSS.}
\]

Ergodicity:
A random process \( x(t) \) is said to be ergodic, if its ensemble averages (statistical averages \( \mathbb{E} \)) mean, autocorrelation) are equal to appropriate time averages.

If \( x(t) \) is a random process, then
\[
\frac{1}{2T} \int_{-T}^{T} x(t) \, dt
\]
is called time average of \( x(t) \) over \( (-T, T) \), and denoted by
\[
\bar{x}_T = \frac{1}{2T} \int_{-T}^{T} x(t) \, dt
\]
If the random process \( x(t) \) has a constant mean,
as \( T \to \infty \), then \( x(t) \) is said to be mean ergodic.

Problem procedure:

Step 1: Find \( \bar{x}_T \)

Step 2: Find \( E[\bar{x}_T] \)

Step 3: \( \text{Var}(\bar{x}_T) = \frac{1}{n^2} \int_c^d \text{C}_{xx}(t) \left(1 - \frac{\text{Var}(x(t))}{n^2}\right) \, dt \)

where

\[
\text{C}_{xx}(t) = E[x(t) x(t+\Delta t)] - E[x(t)] E[x(t+\Delta t)]
\]

Step 4: \( \lim_{T \to \infty} \text{Var}(\bar{x}_T) = 0 \)

Correlation ergodic:

\[
\frac{1}{n^2} \int_c^d x(t) x(t+\Delta t) \, dt = R(\Delta t)
\]

As limit \( T \to \infty \), \( \bar{x}_T \) is correlation ergodic.

\[
\frac{1}{n^2} \int_c^d x(t+\Delta t) x(t) \, dt = R(\Delta t)
\]

If WSS process \( x(t) \) is given by \( x(t) = 10 \cos(100t + \theta) \) where \( \theta \) is uniformly distributed over \((-\pi, \pi)\), prove that \( x(t) \) is correlation ergodic.

Solution:

\[
\text{C}_{xx}(\theta) = \frac{1}{b-a} = \frac{1}{2\pi} = \frac{1}{2\pi}
\]

\[
R_{xx}(\Delta t) = E[x(t) \cdot x(t+\Delta t)]
\]
\[
\begin{align*}
\frac{100}{2} E \left[ \cos (200t + 100\pi + 2\theta) + \cos (100\pi) \right] \\
= 50 E \left[ \cos (200t + 100\pi + 2\theta) + \cos (100\pi) \right] \\
\text{Consider,} \\
E \left[ \cos (200t + 100\pi + 2\theta) \right] = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos (200t + 100\pi + 2\theta) \, d\theta \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos (200t + 100\pi) \, d\theta \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin (200t + 100\pi) \, d\theta \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin (200t) \, d\theta \\
= 0 \\
R_{XX}(t) = 50 \left( 0 + \cos (100\pi) \right) \\
= 50 \cos (100\pi) \\
\frac{1}{2T} \int_{-T}^{T} x(t+t) \, dt = R(t) \\
= \frac{1}{2T} \int_{-T}^{T} 10 \cos (100t + \theta) \, dt \\
= \frac{100}{2T} \int_{-T}^{T} \cos (100t + \theta) \, dt \\
= \frac{100}{2T} \int_{-T}^{T} \cos (100t + \theta) \, dt \\
= \frac{100}{2T} \left[ \frac{1}{2} \left( \cos (100t + \theta) + \cos (100t + 100\pi + \theta) \right) \right]_{-T}^{T} \\
= \frac{100}{2T} \left( \frac{1}{2} \left( \cos (100T + \theta) + \cos (100\pi + 2\theta) \right) + \cos (-100\pi) \right)
\end{align*}
\]
Consider a random variable process,
\[ X(t) = 3 \cos (wt + \Theta) \]
\[ Y(t) = 2 \cos (wt + \Theta - \pi/2) \]
where \( \Theta \) is a random variable uniformly distributed in \((0, 2\pi)\). Prove that
\[ \int R_{XX}(\tau) \cdot R_{YY}(\tau) \geq R_{XY}(\tau) \]

Solution,

Given,
\[ X(t) = 3 \cos (wt + \Theta) \]
\[ Y(t) = 2 \cos (wt + \Theta - \pi/2) \]
\( \Theta \) uniformly distributed in \((0, 2\pi)\)
\[ f(\Theta) = \frac{1}{2\pi} \]
\[ R_{XX}(\tau) = E [X(t) \cdot X(t+\tau)] \]
\[ = E \left[ 3 \cos (wt + \Theta) \cdot 3 \cos (wt + \Theta + \tau) \right] \]
\[ = 9 E \left[ \cos (wt + \Theta) \cdot \cos (wt + \Theta + \tau) \right] \]
\[ = \frac{9}{2} E \left[ \cos (2wt + 2\Theta + \pi) + \cos (\pi) \right] \]
\[ = \frac{9}{2} E \left[ \cos (2wt + 2\Theta + \pi) \right] \]
\[ = 9 E \left[ \cos (2\Theta) \right] \]
\[
\begin{align*}
R_{yy}(\omega) &= 2 \cos \omega \delta(
\omega - \omega_0) \\
R_{xy}(t) &= E\left[ X(t) \cdot Y(t + \tau) \right] \\
&= E\left[ 3 \cos(\omega t + \varphi) \cdot 2 \cos(\omega t + \omega_0 - \frac{\pi}{2}) \right] \\
&= 6 E\left[ \cos(\omega t + \varphi) \cdot \cos(\omega t + \omega_0 - \frac{\pi}{2}) \right] + 6 E\left[ \cos(\omega t + \omega_0 - \frac{\pi}{2}) \right] \\
&= 6 E\left[ \cos\left(\frac{\pi}{2} - \omega t\right) \right] \\
&= 6 E\left[ \cos\left(\omega t + \omega_0 - \frac{\pi}{2}\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0 + \theta\right) \right] + 6 E\left[ \cos\left(\omega t + \omega_0 - \frac{\pi}{2}\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0 + \theta\right) \right] + 6 E\left[ \cos\left(\omega t + \omega_0\right) \right] \\
&= 6 E\left[ \sin\left(\frac{\pi}{2} - \omega t\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0 + \theta\right) \right] + 6 E\left[ \cos\left(\omega t + \omega_0\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0\right) \right] \\
&= 6 E\left[ \sin\left(\omega t + \omega_0\right) \right]
\end{align*}
\]
\[
\begin{align*}
\frac{3}{2} & \left[ \cos(2\omega_0 + \alpha_0 + 2\pi) \cos(\omega_0 + 2\pi) - \cos(\alpha_0) \right] + b \sin \omega_0 \\
& = \frac{3}{2} \sin \omega_0 \\
R_{xy}(\omega) & = 3 \sin \omega_0 \\
R_{xx}(\omega) - R_{yy}(\omega) & = \frac{3}{2} - 2 = \frac{1}{2} \\
\sqrt{R_{xx}(\omega) \cdot R_{yy}(\omega)} & = \sqrt{\frac{1}{2}} \\
& = \frac{\sqrt{2}}{2} \\
R_{xy}(\omega) & = \left| 3 \sin \omega_0 \right| \leq 3 \\
R_{xy}(\omega) & \leq \sqrt{R_{xx}(\omega) \cdot R_{yy}(\omega)}.
\end{align*}
\]
Markov process

Future depends only upon the present but not on past.

If for all \( n \),

\[
P_{x_n = a_n | x_{n-1} = a_{n-1}} = P_{x_n = a_n | x_{n-1} = a_{n-1}}
\]

\[
P_{x_n = a_n | x_{n-1} = a_{n-1}} = \frac{P_{X_n = a_n \cap X_{n-1} = a_{n-1}}}{P_{X_{n-1} = a_{n-1}}}
\]

\[
\sum_{x_n \in S} P_{x_n} = 1
\]

Thus the process \( \{X_n\} \),

\( n = 0, 1, 2, \ldots \) is called Markov Chain.

(i) \( a_1, a_2, \ldots a_n \) are called states.

(ii) \( P_{x_n = a_j | x_{n-1} = a_{i-1}} \) is called one step transition probability from state \( a_i \) to \( a_j \).

(iii) \( P_{x_n = a_j | x_0 = a_{i-1}} \) is called \( n \) step transition probability from state \( a_i \) to \( a_j \).

Note 1:

The TPM of a Markov Chain is a Stochastic matrix since

\( P_{ij} \geq 0 \) and \( \sum P_{ij} = 1 \) (ie) Sum of all the elements of row of the
Note 2:

A stochastic matrix \( P \) is said to be a regular matrix if all the entries of \( P \) (possible integer \( m \)) are positive.

Note 3:

A homogeneous Markov chain is said to be regular if its TPM is regular.

Note 4:

If \( P_{ij} > 0 \) for some \( i \) and \( j \), then every state can be reached from every other state. Hence, the Markov chain is said to be irreducible.

Note 5:

The period \( \text{period} = d_i \) of a state \( i \) is defined as the greatest common divisor of all \( m \), such that \( P_{ij} > 0 \). State \( i \) is said to be periodic with period \( d_i \), if \( d_i > 1 \), and aperiodic if \( d_i = 1 \).
Note 6:
A non-null persistent and called aperiodic state is ergodic.

Note 7:
If a Markov chain is irreducible, all its states are of the same time.
If a Markov chain is finite irreducible, all its states are non-null persistent.

Note 8:
Steady state probability distribution or stationary state distribution of the Markov chain is \( \pi P = \pi \)

Note 9:
To find irreducible nature: \( P^2, P^3, P^4 \ldots \) and note all \( P_{ij} > 0 \) at some \( P^n \).

To find period type; called the powers of \( P \), where \( P^k > 0 \), and find \( \text{gcd of powers} \). 

\( \pi P = \pi \)
Find the nature of the states of the Markov chain using the TPM.

\[ P = \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix} \]

Solution,

Given,

\[ P = \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix} \]

\[ P^2 = P \cdot P = \begin{bmatrix}
0 & 0 & 1 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1/2 & 0 \\
0 & 0 & 1/2 \\
1/2 & 0 & 1/2
\end{bmatrix} \]

\[ P^3 = P \cdot P^2 = \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
0 & 0 & 1 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0 \\
0 & 1 & 0
\end{bmatrix} \]

\[ P^4 = P \cdot P^3 = \begin{bmatrix}
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0 \\
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0 \\
0 & 1 & 0
\end{bmatrix} \]

Here, all vectors can be reached.
Markov chain is irreducible and finite.

⇒ All states are non-null persistent.

1. \( p_{ii} > 0, \ p_{ii}', > 0 \)

⇒ \( \gcd \{ 2, 4, \ldots, 2 \} = 2 \)
⇒ State \((2, 2, \ldots, 2)\) is period 2

\[ p_{22} (2) > 0, \ p_{22} (4) > 0 \]
\[ \gcd \{ 2, 4 \} = 2 \]
⇒ State 2 is period 2

⇒ State 2 is period 2.

Here all states are periodic with period 2.

Here all states are non-null persistent and periodic.

⇒ All states are ergodic.
Three boys A, B, C are throwing a ball to each other, A always throws the ball to B, and B always throws the ball to C. But C is just as likely to throw the ball to B as to A. Find the TPM and classify the states.

Solution:
The TPM is

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}
\]

\[
P^2 = P \cdot P = \begin{bmatrix}
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix}
\]

\[
P^3 = P \cdot P^2 = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{bmatrix}
\]

\[
P^4 = P^2 \cdot P^2 = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{bmatrix}
\]

\[
P^5 = P^3 \cdot P^2 = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{bmatrix}
\]
The markov chain is irreducible if every state communicates with every other state. A markov chain is regular if & only if there exists a positive integer N such that 

\[ p^{(N)} > 0 \]

All states non-null persistent.

\[ p^{(N)} > 0 \]

For any two states, \( \gcd \{ m, n \} = 1 \)

States 'A' is periodic

\[ p_{22} > 0, p_{33} > 0, p_{44} > 0 \]

\[ \gcd \{ 2, 3, 4 \} = 1 \]

States 'B' is periodic

\[ p_{55} > 0, p_{66} > 0, p_{77} > 0 \]

\[ \gcd \{ 2, 3, 4 \} = 1 \]

States 'C' is periodic

All states are periodic.

A man drives a car or takes a train to office each day. He never goes two days in a row by the same mode of transport. He is just likely to drive again as he drove today if he has driven on the previous day. Now suppose that the man starts driving on the first day of the week. The man drives to office.
A man drives a car every day. He has a bus ticket valid for one day. He goes to office by train on the first day. Now he suppose to go to office by car on the second day. He never goes to office by train. But up to the third day he drives one day by train and one day by car. Then he drives only by car. Suppose that he drives by car on the third day. Thereafter he drives only by car. All states are periodic.

\[ P_A(0) > 0 \]
\[ P_A(1) > 0 \]
\[ P_A(2) > 0 \]
\[ P_A(3) > 0 \]
\[ P_A(4) > 0 \]
\[ P_A(5) > 0 \]
\[ P_A(6) > 0 \]
\[ P_A(7) > 0 \]
\[ P_A(8) > 0 \]
\[ P_A(9) > 0 \]
\[ P_A(10) > 0 \]
\[ \text{gcd} \{3, 5, 10\} = 1 \]

The monkey chain is irreducible. All states are non-null persistent.
The transition probability matrix of a Markov chain \( \{X_n\} = 1, 2, 3 \) having three states 1, 2 and 3 is \( P \) :

\[
P = \begin{bmatrix}
0.1 & 0.5 & 0.4 \\
0.6 & 0.2 & 0.2 \\
0.3 & 0.4 & 0.3 \\
\end{bmatrix}
\]

and the initial distribution is \( P^0 = \{0.7, 0.2, 0.1\} \).

Find (i) \( P[X_3 = 3] \)

(ii) \( P[X_2 = 2, X_3 = 3, X_1 = 2, X_0 = 1] \)

Solution:

Given:

\[
P = \begin{bmatrix}
0.1 & 0.5 & 0.4 \\
0.6 & 0.2 & 0.2 \\
0.3 & 0.4 & 0.3 \\
\end{bmatrix}
\]

\[
P^2 = P \cdot P = \begin{bmatrix}
0.45 & 0.31 & 0.24 \\
0.24 & 0.42 & 0.34 \\
0.36 & 0.55 & 0.39 \\
\end{bmatrix}
\]

(i) \( P[X_2 = 3] = \sum_{i=1}^{3} P[X_2 = 3 \mid X_0 = i] \cdot P[X_0 = i] \)

\[
= P[X_2 = 3 \mid X_0 = 1] \cdot P[X_0 = 1] + P[X_2 = 3 \mid X_0 = 2] \cdot P[X_0 = 2] + P[X_2 = 3 \mid X_0 = 3] \cdot P[X_0 = 3]
\]

\[
= P_{13} \cdot P[X_0 = 1] + P_{23} \cdot P[X_0 = 2] + P_{33} \cdot P[X_0 = 3]
\]

\[
= (0.36 \times 0.7) + (0.84 \times 0.2) + (0.34 \times 0.1)
\]


\[
\begin{align*}
&= 0.1824 \times 0.068 + 0.089 \\
&= 0.0279.
\end{align*}
\]

(iii) \( P [X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2] = \) 

\[
P [X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2] P [X_2 = 3, X_1 = 3, X_0 = 2]
\]

\[
= P [X_3 = 2 / X_2 = 3] P [X_2 = 3 / X_1 = 3, X_0 = 2] P [X_1 = 3 / X_0 = 2] P [X_0 = 2]
\]

\[
= P_{32} P_{35} P_{53} P [X_0 = 2]
\]

\[
= (0.1) (0.3) (0.2) (0.2) = 0.0048.
\]

A state \( i \) is said to be recurrent if it returns to state \( i \) is certain, \( F_{ii} = 1 \), and transient if it is uncertain, \( F_{ii} < 1 \).
Consider a markov chain with a states \( S_0, S_1 \) and \( P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \).

(i) Draw the transition diagram.

(ii) Is the state 0 recurrent?

(iii) Is the state 1 transient?

Solution:

\[
\begin{align*}
\text{i)} & \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\
\end{align*}
\]

\[
\begin{align*}
\text{ii) State 0 is recurrent.} & \quad \text{It returns to zero with probability 1.} \\
\end{align*}
\]

\[
\begin{align*}
\text{iii) State 1 is transient.} & \quad \text{It returns to 1 with probability} \\
\end{align*}
\]

Poisson process:

If \( X(t) \) represents the number of occurrences of a certain event in \((0,t)\), then discrete random process \( X(t) \) is called the Poisson process.

(i) \( P(\text{1 occurrence in } (t,t+\Delta t]) = \lambda \Delta t + o(\Delta t) \)

(ii) \( P(\text{0 occurrence in } (t,t+\Delta t]) = e^{-\lambda t} + o(\Delta t) \)

(iii) \( P(\text{2 occurrence in } (t,t+\Delta t]) = 0(\Delta t) \)

(iv) \( X(t) \) is independent.
Second Order Probability function of a homogeneous poisson process.

\[ P_n(t_2) = \frac{e^{-\lambda t} \left( \lambda t \right)^n}{n!} \quad n=0,1,2,\ldots \]

\[ P(x(t_1) = n, x(t_2) = n) = \int P(x(t_1) = n) \cdot P(x(t_2) = n \mid x(t_1) = n) \, dt_1 \]

\[ P(x(t_1) = n, x(t_2) = n) = \frac{e^{-\lambda t_1} \left( \lambda t_1 \right)^n}{n!} \cdot \frac{e^{-\lambda (t_2-t_1)} \left( \lambda (t_2-t_1) \right)^n}{n!} \cdot \frac{1}{(n_2-n)!} \]

Third Order probability function of a homogeneous poisson process:

\[ P(x(t_1) = n, x(t_2) = n_2, x(t_3) = n_3) = \int P(x(t_1) = n) \cdot P(x(t_2) = n_2 \mid x(t_1) = n) \cdot P(x(t_3) = n_3 \mid x(t_1) = n) \, dt_1 \]

\[ = \frac{e^{-\lambda t_1} \left( \lambda t_1 \right)^n}{n!} \cdot \frac{e^{-\lambda (t_2-t_1)} \left( \lambda (t_2-t_1) \right)^n}{n!} \cdot \frac{1}{(n_3-n)!} \cdot \frac{1}{(n_2-n)!} \cdot \frac{1}{(n_3-n_2)!} \]

Mean of a poisson process:

\[ \mu = \lim_{t \to \infty} E[x(t)] = \lim_{t \to \infty} \sum_{n=0}^{\infty} n P_n(t) \]

\[ = \lim_{t \to \infty} \left[ \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \right] \]

\[ = \lim_{t \to \infty} \left[ \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{(n-1)!} \right] \]

\[ = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!} \]

\[ = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} \]

\[ = e^{-\lambda t} \left[ \frac{1}{\lambda} \right]^{(n-1)} \frac{1}{(n-1)!} \]

\[ \mu = \frac{\lambda}{1-\lambda} \]
Auto covariance of the Poisson process:

\[ C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) = E[x(t_1)]E[x(t_2)] \]

\[ = \lambda^2(t_1 t_2) + \lambda t_1 - \lambda t_1 \cdot \lambda t_2 \]

\[ = \lambda^2(t_1 t_2) + \lambda(t_1) - \lambda^2(t_1 t_2) \]

\[ = \lambda(t_1) \]

\[ C_{xx}(t_1, t_2) = \lambda \min(t_1, t_2) \]

Correlation coefficient of the Poisson process:

\[ P_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{\text{Var}(x(t_1)) \cdot \text{Var}(x(t_2))}} \]

\[ = \frac{\lambda(t_1)}{\sqrt{\lambda t_1 \cdot \sqrt{\lambda t_2}}} \]

\[ = \frac{\lambda t_1}{\lambda \sqrt{t_1 t_2}} \]

\[ = \sqrt{\frac{t_1}{t_2}} \quad t_1 \leq t_2 \]

\[ P_{xx}(t_1, t_2) = \sqrt{\frac{t_1}{t_2}} \quad t_1 \leq t_2 \]
Property 1:

Poisson process is a Markov process:

Let us take the conditional probability distribution of $X(t_3)$ given the past values of $X(t_2)$ and $X(t_1)$. Assume that $t_3 > t_2 > t_1$, and $n_3 > n_2 > n_1$.

Consider $P[X(t_3) = n_3 | X(t_2) = n_2, X(t_1) = n_1]$

$$= \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_2) = n_2, X(t_1) = n_1]}$$

$$= \frac{P[X(t_3) = n_3 - n_1, X(t_2) - t_1] e^{-\lambda t_1} \lambda (t_2 - t_1)^{n_2-n_1} \cdot (n_2-n_1)! \cdot (n_3-n_2)!}{P[X(t_2) = n_2, X(t_1) = n_1]}$$

$$= \frac{e^{-\lambda t_1} \lambda (t_2 - t_1)^{n_2-n_1} \cdot (n_2-n_1)! \cdot (n_3-n_2)! \cdot e^{-\lambda t_2} \lambda (t_3 - t_2)^{n_3-n_2} \cdot (n_3-n_2)!}{P[X(t_2) = n_2, X(t_1) = n_1]}$$

$$= \frac{e^{-\lambda t_1} \lambda (t_2 - t_1)^{n_2-n_1} \cdot (n_2-n_1)! \cdot (n_3-n_2)! \cdot e^{-\lambda t_2} \lambda (t_3 - t_2)^{n_3-n_2} \cdot (n_3-n_2)!}{P[X(t_2) = n_2, X(t_1) = n_1]}$$

$$= \frac{e^{-\lambda t_1} \lambda (t_2 - t_1)^{n_2-n_1} \cdot (n_2-n_1)! \cdot (n_3-n_2)! \cdot e^{-\lambda t_2} \lambda (t_3 - t_2)^{n_3-n_2} \cdot (n_3-n_2)!}{P[X(t_2) = n_2, X(t_1) = n_1]}$$
Thus Poisson process is a Markov process.

Property 2:

Sum of two independent Poisson processes is a Poisson process.

Let \( X(t) = X_1(t) + X_2(t) \).

\[
P[X(t) = n] = \sum_{k=0}^{n} P[X_1(t) = k] P[X_2(t) = n-k]
\]

\[
= \sum_{k=0}^{n} \frac{e^{-\lambda_1 t} (\lambda_1 t)^k}{k!} \frac{e^{-\lambda_2 t} (\lambda_2 t)^{n-k}}{(n-k)!}
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^{k+n-k}
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]

\[
= e^{-\lambda t - \lambda_2 t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} t^n
\]
\[
(\lor \in x \Rightarrow P^n \frac{q^{n-x}}{(n-x)!} = e^{-\lambda} \frac{\lambda^n}{n!} (1+\lambda x)^x
\]

\[
\therefore P[x(t)=n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}
\]

\[
\frac{d}{dt} P[x(t)=n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{d}{dt} (\lambda t)^n
\]

\[
= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \lambda^n
\]

\[
= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \lambda^n
\]

Thus, \(x_1(t) + x_2(t)\) is a Poisson process.

Property 2: \(x(t) = x_1(t) + x_2(t)\)

Difference of two independent Poisson processes is not a Poisson process.

Proof:

Let \(x(t) = x_1(t) - x_2(t)\).

\[
E[x(t)] = E[x_1(t) - x_2(t)] = E[x_1(t)] - E[x_2(t)]
\]

\[
= \lambda_1 t - \lambda_2 t
\]

\[
E[x^2(t)] = E[x_1(t) - x_2(t)]^2
\]

\[
= E[x_1^2(t)] + E[x_2^2(t)] - 2E[x_1(t)]E[x_2(t)]
\]

\[
= (\lambda_1^2 + \lambda_2^2) + (\lambda_2^2 + \lambda_2^2) - 2 \lambda_1 \lambda_2 t
\]
\[ = (\lambda_1 + \lambda_2) t + \left( \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \right) t^2 \]

The interval between two successive occurrences of a Poisson process with parameter \( \lambda \) has an exponential distribution with mean \( \frac{1}{\lambda} \).

Proof:

Let \( E_i \) and \( E_{i+1} \) be the two consecutive events.

Let \( T \) be the interval between \( E_i \) and \( E_{i+1} \).

\[ P(T > t) = P[ \text{no event occurs in the interval } (t, t+1) ] = e^{-\lambda t} \]

The distribution function is:

\[ F(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t} \]
\[ f(t) = F'(t) = -e^{-\lambda t}(\lambda - \lambda) = -\lambda e^{-\lambda t} \]

\[ f(t) = \lambda e^{-\lambda t} \] is a PDF of an exponential distribution with mean \( \frac{1}{\lambda} \)

If the no. of occurrences of an event E in an interval of length t, is a Poisson process \( x(t) \) with parameter \( \lambda \) and if each occurrence of E has a constant probability \( P \), being recorded and the recordings are independent of each other, then the no. number \( W(t) \) of the recorded occurrences in \( t \) is also a Poisson process with parameter \( \lambda P \)

Solution:

\[ P[N(t) = n] = \sum_{r=0}^{\infty} P[x(t) = n] \begin{cases} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{cases} \]

\[ = \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^n}{n!} P[q \times r = q] \]

\[ = \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^n}{n!} \frac{q^n}{r^n} \frac{1}{n!} \]

\[ = \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^n}{n!} \frac{q^n}{r^n} \frac{1}{n!} \]

\[ = \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^n}{n!} \frac{q^n}{r^n} \frac{1}{n!} \]

\[ = \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^n}{n!} \frac{q^n}{r^n} \frac{1}{n!} \]
If $X(t)$ is a Poisson process, prove that,

$$P[X(s) = r \mid X(t) = n] = nC_r \left( \frac{s}{t} \right)^r \left( 1 - \frac{s}{t} \right)^{n-r}, \quad 0 \leq r \leq n.$$
If customers arrive at a counter in accordance with the Poisson process with a mean rate of 3 per minute, find the probability that the interval between two consecutive arrivals is

(a) More than 1 minute
(b) Between 1 minute and 2 minutes.
(c) 4 minutes (or less).

**Solution:**

\[ P[T > t] = \int_0^\infty \lambda e^{-\lambda t} \, dt = \int_0^\infty \frac{-\lambda t}{t} \, dt \]

\[ = \left[ -\lambda e^{-\lambda t} \right]_0^\infty = 0 - \left[ -\lambda e^{-\lambda \cdot 1} \right]_0^\infty = \left[ -\lambda e^{-\lambda \cdot 1} \right]_0^\infty = 0 - 0.4978 \]

\[ = 0.4978 \]

\[ P[1 < T < 2] = \int_1^2 \lambda e^{-\lambda t} \, dt \]

\[ = \left[ -\lambda e^{-\lambda t} \right]_1^2 = \left[ -\lambda e^{-\lambda \cdot 2} - (-\lambda e^{-\lambda \cdot 1}) \right] = \lambda e^{-\lambda \cdot 1} - \lambda e^{-\lambda \cdot 2} \]

\[ = 3 \left( e^{-3} - e^{-6} \right) \]
\[- \left[ e^{\frac{t}{2}} - e^{-\frac{t}{2}} \right] \]

We know the probability density function of a Poisson distribution is:

\[ f(t) = \frac{e^{-\lambda t} \lambda^t}{t!} \]

So, the expected value of the Poisson distribution is:

\[ E(T) = \lambda \]

And the variance is:

\[ Var(T) = \lambda \]

The correlation function for a Poisson process is:

\[ R(t) = \lambda e^{-\lambda t} \]

We need to find the probability that the process is strictly increasing at time \( t \).

\[ P(T < t) = \int_0^t \lambda e^{-\lambda t} \, dt \]

\[ = - \left[ \frac{\lambda}{\lambda} e^{-\lambda t} \right]_0^t \]

\[ = - \left[ e^{-\lambda t} - 1 \right]_0^t \]

\[ = e^{-\lambda t} - 1 \]

So, the probability that the process is strictly increasing at time \( t \) is:

\[ P(T < t) = e^{-\lambda t} - 1 \]

Gaussian (Normal) Process:

A real-valued random process \( x(t) \) is said to be a Gaussian process if the random variables \( x(t_1), x(t_2), \ldots, x(t_n) \) are jointly normal for a fixed \( n = 1, 2, \ldots \) and for any set of \( t_i \)'s.
Let \( x(t) \) be a Gaussian random process with \( \mu(x(t)) = 10 \) and \( C_{xx}(t_1, t_2) = 16e^{-|t_1 - t_2|} \).

Find

(i) \( x(10) \leq 8 \)

(ii) \( |x(10) - x(5)| \leq 4 \)

Solution:

\[
C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E[x(t_1) \cdot x(t_2)]
\]

\[
C_{xx}(t_1, t_1) = R_{xx}(t_1, t_1) - E[x(t_1)]^2 = E[x(t_1)^2] - E[x(t_1)]^2
\]

\[
C_{xx}(t_1, t_1) = \text{Var}(x(t_1)) \quad (i)
\]

Given:

\[
C_{xx}(t_1, t_2) = 16e^{-|t_2 - t_1|}
\]

Let \( t_1 = 10 \)

\[
C_{xx}(t_1, t_1) = 16e^{-|t_1 - t_1|} = 16e^0 = 16 \quad (2)
\]

Sub (2) in (i)

\[
\text{Var}[x(t_1)] = C_{xx}(t_1, t_1) = 16
\]

\[
\text{Mean } 10 \text{ and Variance } 16.
\]
(i) \( P(x(b) \leq 8) \)

Let \( Z = \frac{x - \mu}{\sigma} \)

\[ Z = \frac{x - 10}{\sqrt{16}} \]

\[ Z = \frac{x - 10}{4} \]

\[ P(x(0) \leq 8) = P\left( Z \leq \frac{8 - 10}{4} \right) \]

\[ = P\left( Z \leq -0.5 \right) \]

\[ = 0.5 - P(0 \leq Z \leq 0.5) \]

\[ = 0.5 - 0.1915 \]

\[ = 0.3085 \]

(ii) \( P(x(b) - x(10) \leq 1) \)

Let \( U = x(b) - x(10) \).

\[ E[U] = E[x(b)] - E[x(10)] \]

\[ = 10 - 5 \]

\[ = 5 \]

\[ E[U^2] = E[x(b)^2] - E[x(10)^2] \]

\[ = E\left[\left(x(b) - x(10)\right)^2\right] \]

\[ = E\left[x(b)^2 - 2x(b)x(10) + x(10)^2\right] \]

\[ = E[x(b)^2] + E[x(10)^2] - 2E[x(b)]E[x(10)] \]

\[ = E[x(b)^2] + E[x(10)^2] - 2 \cdot 5 \cdot 5 \]

\[ = E[x(b)^2] + E[x(10)^2] - 50 \]
\[ \text{Var}(v) = E\left[v^2\right] - [E[v]]^2 \]

\[ = E\left[\chi^2(n)\right] + E\left[\chi^2(b)\right] - 2 \text{cov}(10, b) \]

\[ = \text{cov}(10, 10) + \text{cov}(b, b) - 2 \text{cov}(10, b) \]

\[ = 1b + 1b - 32 \phi \]

\[ = 0 \Phi 32 - 32 (0.0183) \]

\[ = 32 - 0.5861 \]

\[ = 31.413 \]

\[ \sigma_v^2 = 21.413 \]

\[ \sigma_v = 5.604 \]

\[ P\left[\chi^2(10) - \chi^2(6) \leq 4^2\right] = P\left[101 \leq 4^2\right] \]

\[ = P\left[\frac{-4 \leq Y \leq 4}{\sqrt{5.604}}\right] \]

\[ Z = \frac{Y - \mu}{\sigma} = \frac{Y - 0}{\sqrt{5.604}} \]

\[ U = 4 \]

\[ Z = \frac{4 - 0}{\sqrt{5.604}} = \frac{4}{5.604} \]

\[ P\left[-1.04 \leq Z \leq 1.04\right] \]

\[ = P\left[-0.136 \leq Z \leq 0.136\right] \]

\[ = P\left[0.136 \leq Z \leq 0.136\right] \]
\[ P \left[ 0 < z < 0.7136 \right] = \frac{1}{\sqrt{2\pi}} \int_{0}^{0.7136} e^{-\frac{1}{2}t^2} dt \\
\]

Suppose \( X(t) \) is a normal process with mean \( \mu(t) = 0 \) and \( \sigma(t) = \frac{1}{2}t \).

Find (i) \( X(s) \leq 2 \)

(ii) \( |X(s) - X(t)| \leq 1 \)

Solution:

\[ C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)X(t_2)] \]

\[ C_{XX}(t_1, t_1) = R_{XX}(t_1, t_1) - E[X(t_1)^2] \]

\[ C_{XX}(t_1, t_2) = \text{var} (X(t_1)) - E[X(t_1)]^2 \]

\[ C_{XX}(t_1, t_2) = -0.2(t_1 - t_2) \]

Put \( t = t_1 = t_2 \)

\[ C_{XX}(t, t) = \mu e^{-0.2t} \]

\[ C_{XX}(t, t) = \mu e^{-0.210} \]

\[ C_{XX}(t, t) = 4 \]

Substitute (2) in (1)
To find \( P\{X(5) \leq 2\} \):

Let 
\[
Z = \frac{X - \mu}{\sigma} = \frac{x - 3}{\sqrt{4}}
\]

\[
Z = \frac{x - 3}{2}
\]

\[P\{X(5) \leq 2\} = P\left[Z \leq \frac{2 - 3}{2}\right]\]

\[= P\{Z \leq -0.5\}\]

\[= 0.5 - P\{0 \leq Z \leq 0.5\}\]

\[= 0.5 - 0.1915\]

\[= 0.3085\]

For \( P\{X(8) - X(5) \leq 1\} \):

Let 
\[
U = X(8) - X(5)
\]

\[E[U] = E[X(8)] - E[X(5)] = 0\]

\[E[U^2] = E[(X(8) - X(5))^2] = E[X(8]^2 - 2E[X(8)X(5)] + E[X(5)]\]

\[= E[X(8)^2] - 2E[X(8)]E[X(5)] + E[X(5)]\]

\[= E[X(8)^2] - 2E[X(8)]E[X(5)]\]
\[ \begin{align*}
E [x^2(x)] + E [x^2(y)] - 2 \text{ cov}(x, y) \\
= \text{ cov}(x, x) + \text{ cov}(x, y) - 2 \text{ cov}(x, y) \\
= x + x - 8e^{-0.6}
\end{align*} \]

\[ = 8 - 8e^{-0.6} \]

\[ = 8(1 - e^{-0.6}) \]

\[ \sigma_x = 3.609 \]

\[ \sigma_y = 1.899 \]

\[ P\left\{ |x(1) - x(5)| \leq 1 \right\} = P\left\{ |101| \leq 1 \right\} \]

Let \[ Z = \frac{U - \mu}{\sigma} = \frac{U - 0}{1.899} \]

Put \[ U = -1 \Rightarrow Z = \frac{-1}{1.899} \approx -0.5265 \]

Put \[ U = 1 \Rightarrow Z = \frac{1}{1.899} \approx 0.5265 \]

\[ P\left\{ |x(1) - x(5)| \leq 1 \right\} = P\left\{ -0.5265 \leq Z \leq -0.5265 \right\} \]

\[ = 2 \Phi(0.5265) = 2 \Phi(0.2099) \]

\[ = 0.4038 \]

\[ P\left\{ |x(1) - x(5)| \leq 1 \right\} = 0.4038 \]
Random Telegram process:

Random Telegram process is a discrete random process \( x(t) \), satisfies the following conditions:

(i) \( x(t) \) assumes only two values -1 and 1.

(ii) \( P[x(0) = 1] = \frac{1}{2} = P[x(0) = -1] \)

(iii) The no. of level transitions (or) flips \( N(t) \) in the interval length \( t \) follows Poisson process.

\[
P[N(t) = r] = e^{-\frac{\lambda t}{2}} \frac{\lambda t^r}{r!} \quad r = 1, 2, 3, ...
\]

Sine wave process:

A sine wave random process is represented as \( x(t) = A \sin(\omega t + \theta) \), where amplitude \( A \) (or) frequency \( \omega \) (or) phase \( \theta \) (or) any combination of these three may be removed.

For the sine wave process \( x(t) = Y \cos \omega_0 t \), \( -\infty < t < \infty \), \( \omega_0 = \text{Constant} \). The amplitude \( Y \) is a random variable with uniform distribution in the interval 0 to 1. Check whether the process is stationary or not.
Solution:
Given $Y$ is uniformly distributed in the interval $(0, 1)$,

$$F(y) = \frac{1}{1-y} = 1$$

$$\mathbb{E}[x(t)] = \int x(t) \cdot F_Y(dy)$$

$$= \int y \cos \omega t \, dy$$

$$= \cos \omega t \int y \, dy$$

$$= \cos \omega t \left( \frac{y^2}{2} \right)_{0}^{1}$$

$$= \cos \omega t \left( \frac{1}{2} - \frac{0}{2} \right)$$

$$= \frac{1}{2} \cos \omega t$$

Since the mean is time-dependent,

Thus the process is not stationary.