Concept and Definition

Operations research signifies research on operations. It is the organized application of modern science, mathematics and computer techniques to complex military, government, business or industrial problems arising in the direction and management of large systems of men, material, money and machines. The purpose is to provide the management with explicit quantitative understanding and assessment of complex situations to have sound basics for arriving at best decisions.

Operations research seeks the optimum state in all conditions and thus provides optimum solution to organizational problems.

Definition: OR is a scientific methodology – analytical, experimental and quantitative – which by assessing the overall implications of various alternative courses of action in a management system provides an improved basis for management decisions.

Characteristics of OR (Features)

The essential characteristics of OR are

1. **Inter-disciplinary team approach** – The optimum solution is found by a team of scientists selected from various disciplines.
2. **Wholistic approach to the system** – OR takes into account all significant factors and finds the best optimum solution to the total organization.
3. **Imperfectness of solutions** – Improves the quality of solution.
4. **Use of scientific research** – Uses scientific research to reach optimum solution.
5. **To optimize the total output** – It tries to optimize by maximizing the profit and minimizing the loss.

Phases of OR

OR study generally involves the following major phases

1. Defining the problem and gathering data
2. Formulating a mathematical model
3. Deriving solutions from the model

4. Testing the model and its solutions
5. Preparing to apply the model
6. Implementation

1. Defining the problem and gathering data

- The first task is to study the relevant system and develop a well-defined statement of the problem. This includes determining appropriate objectives, constraints, interrelationships and alternative course of action.
- The OR team normally works in an **advisory capacity**. The team performs a detailed technical analysis of the problem and then presents recommendations to the management.
- Ascertaining the appropriate **objectives** is very important aspect of problem definition. Some of the objectives include maintaining stable price, profits, increasing the share in market, improving work morale etc.
- OR team typically spends huge amount of time in gathering relevant data.
  - To gain accurate understanding of problem
  - To provide input for next phase.
- OR teams uses Data mining methods to search large databases for interesting patterns that may lead to useful decisions.

2. Formulating a mathematical model

This phase is to reformulate the problem in terms of mathematical symbols and expressions. The mathematical model of a business problem is described as the system of equations and related mathematical expressions. Thus

1. **Decision variables** ($x_1, x_2 \ldots x_n$) – 'n' related quantifiable decisions to be made.
2. **Objective function** – measure of performance (profit) expressed as mathematical function of decision variables. For example $P=3x_1 +5x_2 + \ldots + 4x_n$
3. **Constraints** – any restriction on values that can be assigned to decision variables in terms of inequalities or equations. For example $x_1 +2x_2 \geq 20$
4. **Parameters** – the constant in the constraints (right hand side values)

Principles of Modeling

The model building and their uses both should be consciously aware of the following ten principles

1. Do not build up a complicated model when simple one will suffice
2. Beware of molding the problem to fit the technique
3. The deduction phase of modeling must be conducted rigorously
4. Models should be validated prior to implementation
5. A model should never be taken too literally
6. A model should neither be pressed to do nor criticized for failing to do that for which it was never intended
7. Beware of over-selling a model
8. Some of the primary benefits of modeling are associated with the process of developing the model
9. A model cannot be any better than the information that goes into it
10. Models cannot replace decision makers

Simplifications of OR Models

While constructing a model, two conflicting objectives usually strike in our mind

1. The model should be as accurate as possible
2. It should be as easy as possible in solving

Besides, the management must be able to understand the solution of the model and must be capable of using it. So the reality of the problem under study should be simplified to the extent when there is no loss of accuracy. The model can be simplified by

- Omitting certain variable
- Changing the nature of variables
- Aggregating the variables
- Changing the relationship between variables

Modifying the constraints, etc

Introduction to Linear Programming

A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where a_1, a_2, …, a_n are constants and $x_1, x_2 \dots x_n$ are variables. The term Programming refers to the process of determining a particular program or plan of action. So Linear Programming (LP) is one of the most important optimization (maximization / minimization) techniques developed in the field of Operations Research (OR).

The methods applied for solving a linear programming problem are basically simple problems; a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n-variables ($x_1, x_2 \dots x_n$), at least one of them is non-zero, can be obtained if there are exactly *n* relations. When the number of relations is greater than or less than *n*, a unique solution does not exist but a number of trial solutions can be found.

In various practical situations, the problems are seen in which the number of relations is not equal to the number of the number of variables and many of the relations are in the form of inequalities (≤ or ≥) to maximize or minimize a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problem (LPP).

Definition – The general LPP calls for optimizing (maximizing / minimizing) a linear function of variables called the '**Objective function**' subject to a set of linear equations and / or inequalities called the '**Constraints**' or '**Restrictions**'.

General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for $x_1, x_2 \dots x_n$ so as to maximize or minimize

$$Z = c_1x_1 + c_2x_2 + \dots\dots\dots\dots + c_nx_n$$

subject to restrictions

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots + a_1nx_n \ (\le \text{ or } \ge)\ b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots + a_2nx_n \ (\le \text{ or } \ge)\ b_2$$

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$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \ (\leq \text{or} \geq)\ b_m$

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Where

Z = value of overall measure of performance

x_j = level of activity (for j = 1, 2, ..., n)

c_j = increase in Z that would result from each unit increase in level of activity j

b_i = amount of resource i that is available for allocation to activities (for i = 1,2, …, m)

a_{ij} = amount of resource i consumed by each unit of activity j

Resource	Resource usage per unit of activity	Amount of resource available
	Activity	
	1 2 …………………….. n	
1	a_{11} a_{12} ……………………. a_{1n}	b_1
2	a_{21} a_{22} ……………………. a_{2n}	b_2
.	.	.
.	.	.
.	.	.
m	a_{m1} a_{m2} ……………………. a_{mn}	b_m
Contribution to Z per unit of activity	c_1 c_2 ………………………. c_n	

Data needed for LP model

- The level of activities $x_1, x_2 \ldots\ldots x_n$ are called **decision variables**.
- The values of the c_j, b_i, a_{ij} (for i=1, 2 … m and j=1, 2 … n) are the **input constants** for the model. They are called as **parameters** of the model.
- The function being maximized or minimized $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is called **objective function**.
- The restrictions are normally called as **constraints**. The constraint $a_{i1}x_1 + a_{i2}x_2 \ldots a_{in}x_n$ are sometimes called as **functional constraint** (L.H.S constraint). $x_j \geq 0$ restrictions are called **non-negativity constraint**.

Advantages of Linear Programming Techniques

1. It helps us in making the optimum utilization of productive resources.
2. The quality of decisions may also be improved by linear programming techniques.
3. Provides practically solutions.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique.

Limitations of Linear Programming

Some limitations are associated with linear programming techniques

1. In some problems, objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.
2. There is no guarantee of getting integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.
3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
4. Sometimes large scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.
5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.
6. Linear programming deals with only single objective, whereas in real life situation problems come across with multi objectives. Goal programming and multi-objective programming deals with such problems.

Formulation of LP Problems

Example 1

A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires 1 minute of processing time on G and 2 minutes on H; type B requires 1 minute on G and 1 minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Solution

Let x_1 be the number of products of type A

x_2 be the number of products of type B

After understanding the problem, the given information can be systematically arranged in the form of the following table.

	Type of products (minutes)		
Machine	Type A (x_1 units)	Type B (x_2 units)	Available time (mins)
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, 2 x_1 will be the profit on selling x_1 units of type A. similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of type B is given by

Maximize $Z = 2x_1 + 3x_2$ (**objective function**)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + 3x_2$.

But, machine G is not available for more than 6 hours 40 minutes (400 minutes). Therefore,

$x_1 + x_2 \leq 400$ (**first constraint**)

Also, the machine H is available for 10 hours (600 minutes) only, therefore,

$2x_1 + 3x_2 \leq 600$ (**second constraint**)

Since it is not possible to produce negative quantities

$x_1 \geq 0$ and $x_2 \geq 0$ (**non-negative restrictions**)

Hence

Maximize $Z = 2x_1 + 3x_2$

Subject to restrictions

$x_1 + x_2 \leq 400$

$2x_1 + 3x_2 \leq 600$

and non-negativity constraints

$x_1 \geq 0$, $x_2 \geq 0$

Example 2

A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields a profit of Rs 45. Product B requires 20 quintals of raw materials, 15 man hours and yields a profit of Rs 80. Formulate the LPP.

Solution

Let x_1 be the number of units of product A

x_2 be the number of units of product B

	Product A	Product B	Availability
Raw materials	5	20	400
Man hours	10	15	450
Profit	Rs 45	Rs 80	

Hence

Maximize $Z = 45x_1 + 80x_2$

Subject to

$5x_1 + 20x_2 \leq 400$

$10x_1 + 15x_2 \leq 450$

$x_1 \geq 0$, $x_2 \geq 0$

Example 3

A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is given the required processing time in minutes for each machine on each product.

	Products		
Machine	A	B	C
X	4	3	5
Y	2	2	4

Machine X and Y have 2000 and 2500 machine minutes. The firm must manufacture 100 A's, 200 B's and 50 C's type, but not more than 150 A's.

Solution

Let x_1 be the number of units of product A, x_2 be the number of units of product B

x_3 be the number of units of product C

	Products			
Machine	A	B	C	Availability
X	4	3	5	2000
Y	2	2	4	2500
Profit	3	2	4	

Max $Z = 3x_1 + 2x_2 + 4x_3$

Subject to

$4x_1 + 3x_2 + 5x_3 \leq 2000$

$2x_1 + 2x_2 + 4x_3 \leq 2500$

$100 \leq x_1 \leq 150$

$x_2 \geq 200$

$x_3 \geq 50$

Example 4

A company owns 2 oil mills A and B which have different production capacities for low, high and medium grade oil. The company enters into a contract to supply oil to a firm every week with 12, 8, 24 barrels of each grade respectively. It costs the company Rs 1000 and Rs 800 per day to run the mills A and B. On a day A produces 6, 2, 4 barrels of each grade and B produces

2, 2, 12 barrels of each grade. Formulate an LPP to determine number of days per week each mill will be operated in order to meet the contract economically.

Solution

Let x_1 be the no. of days a week the mill A has to work

x_2 be the no. of days per week the mill B has to work

Grade	A	B	Minimum requirement
Low	6	2	12
High	2	2	8
Medium	4	12	24
Cost per day	Rs 1000	Rs 800	

Minimize $Z = 1000x_1 + 800\,x_2$

Subject to

$6x_1 + 2x_2 \geq 12$

$2x_1 + 2x_2 \geq 8$

$4x_1 + 12x_2 \geq 24$

$x_1 \geq 0$, $x_2 \geq 0$

Example 5

A company has 3 operational departments weaving, processing and packing with the capacity to produce 3 different types of clothes that are suiting, shirting and woolen yielding with the profit of Rs. 2, Rs. 4 and Rs. 3 per meters respectively. 1m suiting requires 3mins in weaving 2 mins in processing and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving 1 min in processing and 3 mins in packing while 1m of woolen requires 3 mins in each department. In a week total run time of each department is 60, 40 and 80 hours for weaving, processing and packing department respectively. Formulate a LPP to find the product to maximize the profit.

Solution

Letx_1 be the number of units of suiting , x_2 be the number of units of shirting

x_3 be the number of units of woolen

	Suiting	Shirting	Woolen	Available time
Weaving	3	4	3	60
Processing	2	1	3	40
Packing	1	3	3	80

Profit	2	4	3	

Maximize $Z = 2x_1 + 4x_2 + 3x_3$
Subject to

$3x_1 + 4x_2 + 3x_3 \leq 60$

$2x_1 + 1x_2 + 3x_3 \leq 40$

$x_1 + 3x_2 + 3x_3 \leq 80$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Example 6

ABC Company produces both interior and exterior paints from 2 raw materials m1 and m2. The following table produces basic data of problem.

	Exterior paint	Interior paint	Availability
M1	6	4	24
M2	1	2	6
Profit per ton	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also maximum daily demand for interior paint is 2 tons. Formulate LPP to determine the best product mix of interior and exterior paints that maximizes the daily total profit.

Solution

Let x_1 be the number of units of exterior paint
x_2 be the number of units of interior paint
Maximize $Z = 5x_1 + 4x_2$
Subject to

$6x_1 + 4x_2 \leq 24$
$x_1 + 2x_2 \leq 6$
$x_2 - x_1 \leq 1$

$x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

b) The maximum daily demand for exterior paint is atmost 2.5 tons

$x_1 \leq 2.5$

c) Daily demand for interior paint is atleast 2 tons

$x_2 \geq 2$

d) Daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

$x_2 > x_1 + 1$

Example 7

A company produces 2 types of hats. Each hat of the I type requires twice as much as labour time as the II type. The company can produce a total of 500 hats a day. The market limits daily sales of I and II types to 150 and 250 hats. Assuming that the profit per hat are Rs.8 for type A and Rs. 5 for type B. Formulate a LPP models in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution

Let x_1 be the number of hats produced by type A

Let x_2 be the number of hats produced by type B

Maximize $Z = 8x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 500$ (labour time)

$x_1 \leq 150$

$x_2 \leq 250$

$x_1 \geq 0,\ x_2 \geq 0$

Example 8

A manufacturer produces 3 models (I, II and III) of a certain product. He uses 2 raw materials A and B of which 4000 and 6000 units respectively are available. The raw materials per unit of 3 models are given below.

Raw materials	I	II	III
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and thrice that of model III. The entire labour force of factory can produce an equivalent of 2500 units of model I. A model survey indicates that the minimum demand of 3 models is 500, 500 and 375 units respectively. However the ratio of number of units produced must be equal to 3:2:5. Assume that profits per unit of model are 60, 40 and 100 respectively. Formulate a LPP.

Solution

Let x_1 be the number of units of model I ,x_2 be the number of units of model II

x_3 be the number of units of model III

Raw materials	I	II	III	Availability
A	2	3	5	4000
B	4	2	7	6000
Profit	60	40	100	

$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$ [Labour time]

$x_1 \geq 500, x_2 \geq 500, x_3 \geq 375$ [Minimum demand]

The given ratio is $x_1: x_2: x_3 = 3: 2: 5$

$x_1 / 3 = x_2 / 2 = x_3 / 5 = k$

$x_1 = 3k; x_2 = 2k; x_3 = 5k$

$x_2 = 2k \rightarrow k = x_2 / 2$

Therefore $x_1 = 3\, x_2 / 2 \rightarrow 2x_1 = 3x_2$

Similarly $2x_3 = 5x_2$

Maximize $Z= 60x_1 + 40x_2 + 100x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \leq 4000$

$4x_1 + 2x_2 + 7x_3 \leq 6000$

$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$

$2\, x_1 = 3x_2$

$2\, x_3 = 5x_2$

and $x_1 \geq 500, x_2 \geq 500, x_3 \geq 375$

Example 9

A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the table.

Food Type	Yield/unit			Cost/Unit Rs
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65

Minimum Requirement	800	200	700	

Formulate the LP for the problem.

Solution

Let x_1 be the number of units of food type 1 , x_2 be the number of units of food type 2

x_3 be the number of units of food type 3, x_4 be the number of units of food type 4

Minimize $Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$

Subject to

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$

$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$

$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

Graphical Solution Procedure

The graphical solution procedure

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph as each one will geometrically represent a straight line.
3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is '≤' then the region below the line lying in the first quadrant is shaded. Similarly for '≥' the region above the line is shaded. The points lying in the common region will satisfy the constraints. This common region is called **feasible region.**
4. Choose the convenient value of Z and plot the objective function line.
5. Pull the objective function line until the extreme points of feasible region.
    a. In the maximization case this line will stop far from the origin and passing through at least one corner of the feasible region.
    b. In the minimization case, this line will stop near to the origin and passing through at least one corner of the feasible region.
6. Read the co-ordinates of the extreme points selected in step 5 and find the maximum or minimum value of Z.

Definitions

1. **Solution** – Any specification of the values for decision variable among (x_1, x_2… x_n) is called a solution.
2. **Feasible solution** is a solution for which all constraints are satisfied.
3. **Infeasible solution** is a solution for which atleast one constraint is not satisfied.
4. **Feasible region** is a collection of all feasible solutions.
5. **Optimal solution** is a feasible solution that has the most favorable value of the objective function.
6. **Most favorable value** is the largest value if the objective function is to be maximized, whereas it is the smallest value if the objective function is to be minimized.
7. **Multiple optimal solution** – More than one solution with the same optimal value of the objective function.
8. **Unbounded solution** – If the value of the objective function can be increased or decreased indefinitely such solutions are called unbounded solution.
9. **Feasible region** – The region containing all the solutions of an inequality
10. **Corner point feasible solution** is a solution that lies at the corner of the feasible region.

Example problems

Example 1

Solve $3x + 5y < 15$ graphically

Solution

Write the given constraint in the form of equation i.e. $3x + 5y = 15$

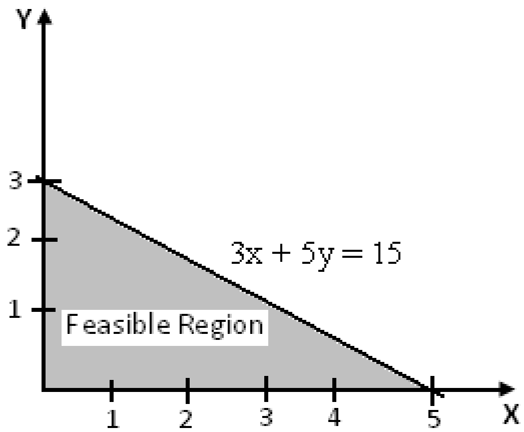
Put x=0 then the value y=3

Put y=0 then the value x=5

Therefore the coordinates are (0, 3) and (5, 0). Thus these points are joined to form a straight line as shown in the graph.

Put x=0, y=0 in the given constraint then

0<15, the condition is true. (0, 0) is solution nearer to origin. So shade the region below the line, which is the feasible region.



Example 2

Solve 3x + 5y >15

Solution

Write the given constraint in the form of equation i.e. 3x + 5y = 15

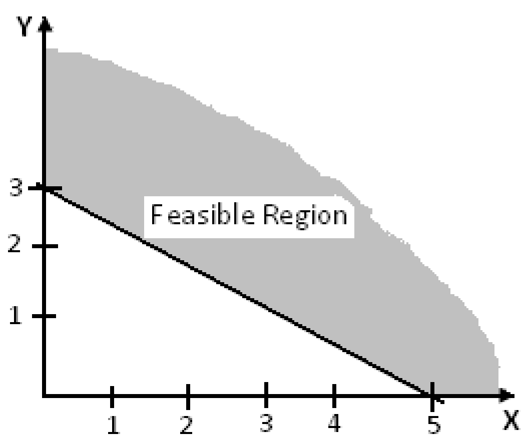
Put x=0, then y=3

Put y=0, then x=5

So the coordinates are (0, 3) and (5, 0)

Put x =0, y =0 in the given constraint, the condition turns out to be false i.e. 0 > 15 is false.

So the region does not contain (0, 0) as solution. The feasible region lies on the outer part of the line as shown in the graph.



Example 3

Max $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $4x_1 + 2x_2 \leq 40$, written in a form of equation

$4x_1 + 2x_2 = 40$

Put $x_1 = 0$, then $x_2 = 20$

Put $x_2 = 0$, then $x_1 = 10$

The coordinates are (0, 20) and (10, 0)

The second constraint $2x_1 + 4x_2 \leq 32$, written in a form of equation

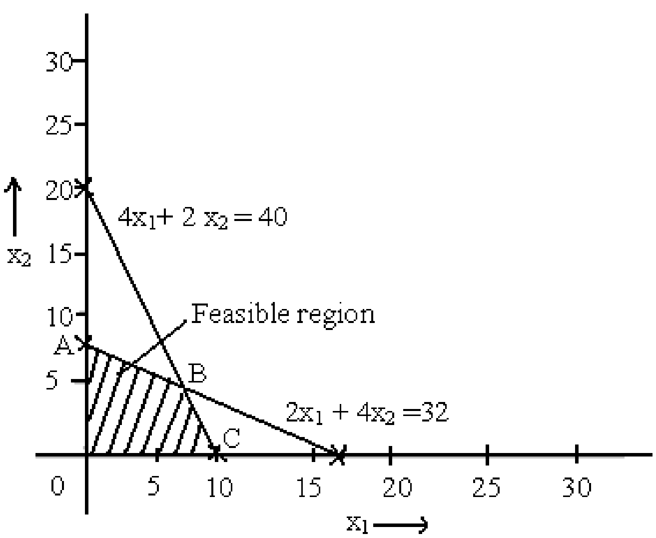
$2x_1 + 4x_2 = 32$

Put $x_1 = 0$, then $x_2 = 8$

Put $x_2 = 0$, then $x_1 = 16$

The coordinates are (0, 8) and (16, 0)

The graphical representation is



The corner points of feasible region are A, B and C. So the coordinates for the corner points are

A (0, 8)

B (8, 4) (Solve the two equations $4x_1 + 2x_2 = 40$ and $2x_1 + 4x_2 = 32$ to get the coordinates)

C (10, 0)

We know that Max $Z = 80x_1 + 55x_2$

At A (0, 8)

$Z = 80(0) + 55(8) = 440$

At B (8, 4)

$Z = 80(8) + 55(4) = 860$

At C (10, 0)

$Z = 80(10) + 55(0) = 800$

The maximum value is obtained at the point B. Therefore Max Z = 860 and $x_1 = 8$, $x_2 = 4$

Example 4

Minimize $Z = 10x_1 + 4x_2$

Subject to

$3x_1 + 2x_2 \geq 60$

$7x_1 + 2x_2 \geq 84$

$3x_1 + 6x_2 \geq 72$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $3x_1 + 2x_2 \geq 60$, written in a form of equation

$3x_1 + 2x_2 = 60$

Put $x_1 = 0$, then $x_2 = 30$

Put $x_2 = 0$, then $x_1 = 20$

The coordinates are (0, 30) and (20, 0)

The second constraint $7x_1 + 2x_2 \geq 84$, written in a form of equation

$7x_1 + 2x_2 = 84$

Put $x_1 = 0$, then $x_2 = 42$

Put $x_2 = 0$, then $x_1 = 12$

The coordinates are (0, 42) and (12, 0)

The third constraint $3x_1 + 6x_2 \geq 72$, written in a form of equation

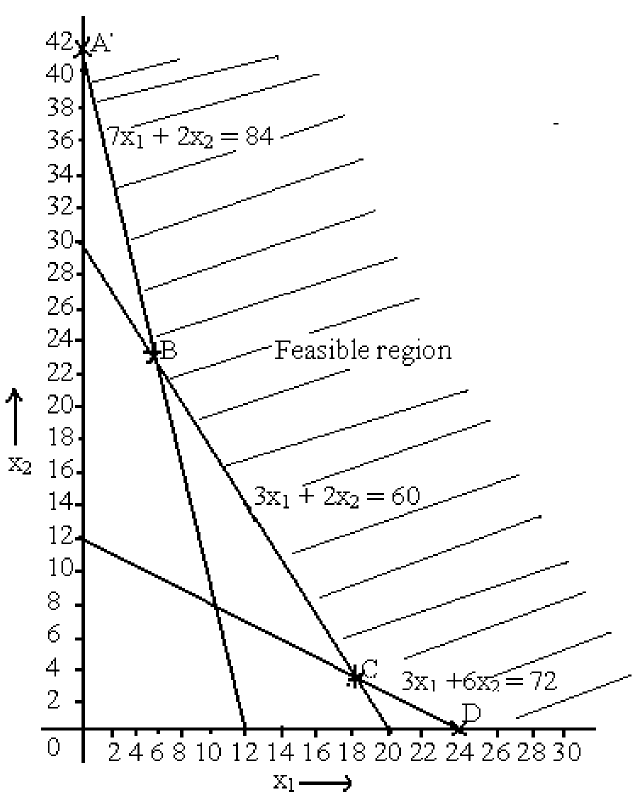
$3x_1 + 6x_2 = 72$

Put $x_1 = 0$, then $x_2 = 12$

Put $x_2 = 0$, then $x_1 = 24$

The coordinates are (0, 12) and (24, 0)

The graphical representation is



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 42)

B (6, 21) (Solve the two equations $7x_1 + 2x_2 = 84$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

C (18, 3) Solve the two equations $3x_1 + 6x_2 = 72$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

D (24, 0)

We know that Min $Z = 10x_1 + 4x_2$

At A (0, 42)

$Z = 10(0) + 4(42) = 168$

At B (6, 21)

$Z = 10(6) + 4(21) = 144$

At C (18, 3)

$Z = 10(18) + 4(3) = 192$

At D (24, 0)

$Z = 10(24) + 4(0) = 240$

The minimum value is obtained at the point B. Therefore Min Z = 144 and $x_1 = 6$, $x_2 = 21$

Example 5

A manufacturer of furniture makes two products – chairs and tables. Processing of this product is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs 2 and Rs 10 respectively. What should be the daily production of each of two products?

Solution

Let x_1 denotes the number of chairs

Let x_2 denotes the number of tables

	Chairs	Tables	Availability
Machine A	2	5	16
Machine B	6	0	30
Profit	Rs 2	Rs 10	

LPP

Max $Z = 2x_1 + 10x_2$

Subject to

$2x_1 + 5x_2 \leq 16$

$6x_1 + 0x_2 \leq 30$

$x_1 \geq 0$, $x_2 \geq 0$

Solving graphically

The first constraint $2x_1 + 5x_2 \leq 16$, written in a form of equation

$2x_1 + 5x_2 = 16$

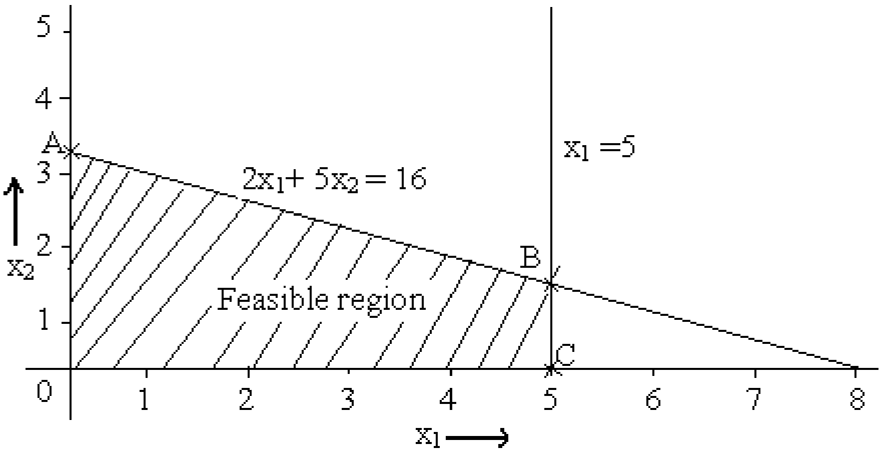
Put $x_1 = 0$, then $x_2 = 16/5 = 3.2$

Put $x_2 = 0$, then $x_1 = 8$

The coordinates are (0, 3.2) and (8, 0)

The second constraint $6x_1 + 0x_2 \leq 30$, written in a form of equation

$6x_1 = 30 \rightarrow x_1 = 5$



The corner points of feasible region are A, B and C. So the coordinates for the corner points are

A (0, 3.2)

B (5, 1.2) (Solve the two equations $2x_1 + 5x_2 = 16$ and $x_1 = 5$ to get the coordinates)

C (5, 0)

We know that Max $Z = 2x_1 + 10x_2$

At A (0, 3.2)

Z = 2(0) + 10(3.2) = 32

At B (5, 1.2)

Z = 2(5) + 10(1.2) = 22

At C (5, 0)

Z = 2(5) + 10(0) = 10

Max Z = 32 and $x_1 = 0$, $x_2 = 3.2$

The manufacturer should produce approximately 3 tables and no chairs to get the max profit.

Special Cases in Graphical Method

Multiple Optimal Solution

Example 1

Solve by using graphical method

Max Z = $4x_1 + 3x_2$

Subject to

$4x_1 + 3x_2 \le 24$

$x_1 \le 4.5$

$x_2 \le 6$

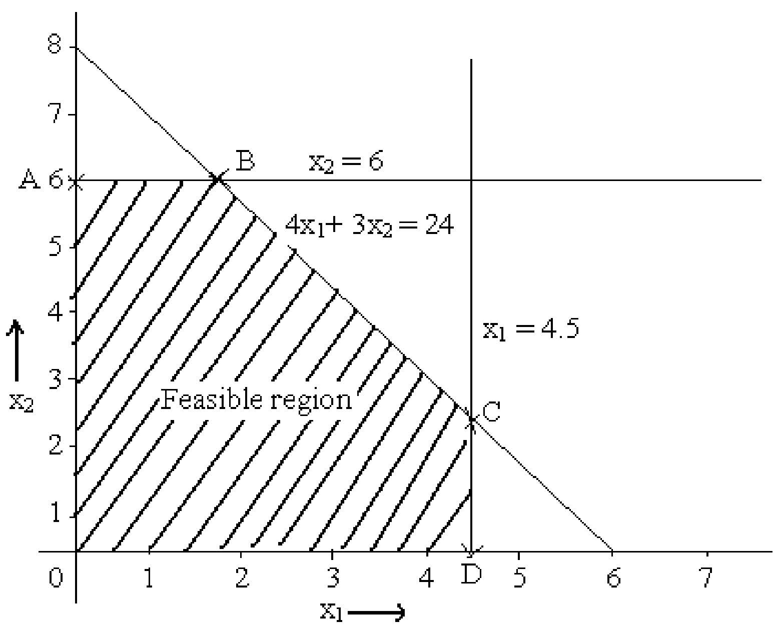
$x_1 \ge 0$, $x_2 \ge 0$

Solution

The first constraint $4x_1 + 3x_2 \le 24$, written in a form of equation
$4x_1 + 3x_2 = 24$
Put $x_1 = 0$, then $x_2 = 8$
Put $x_2 = 0$, then $x_1 = 6$
The coordinates are (0, 8) and (6, 0)

The second constraint $x_1 \le 4.5$, written in a form of equation
$x_1 = 4.5$

The third constraint $x_2 \le 6$, written in a form of equation
$x_2 = 6$



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 6)

B (1.5, 6) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_2 = 6$ to get the coordinates)

C (4.5, 2) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_1 = 4.5$ to get the coordinates)

D (4.5, 0)

We know that Max $Z = 4x_1 + 3x_2$
At A (0, 6)
$Z = 4(0) + 3(6) = 18$

At B (1.5, 6)
$Z = 4(1.5) + 3(6) = 24$

At C (4.5, 2)
$Z = 4(4.5) + 3(2) = 24$

At D (4.5, 0)
$Z = 4(4.5) + 3(0) = 18$
Max Z = 24, which is achieved at both B and C corner points. It can be achieved not only at B and C but every point between B and C. Hence the given problem has multiple optimal solutions.

3.4.2 No Optimal Solution

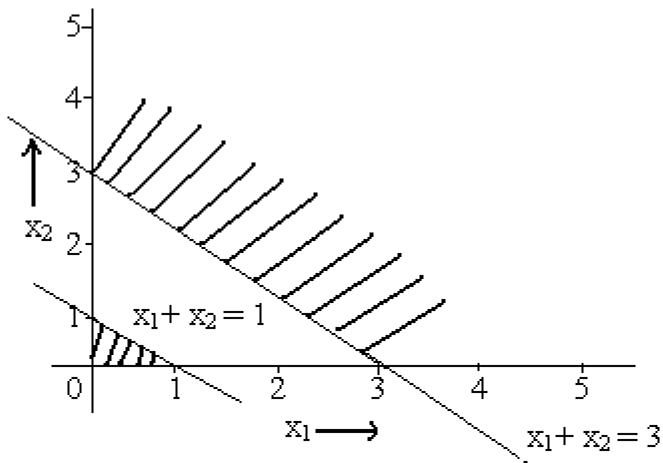
Example 1
Solve graphically
Max $Z = 3x_1 + 2x_2$
Subject to

$x_1 + x_2 \leq 1$
$x_1 + x_2 \geq 3$
$x_1 \geq 0$, $x_2 \geq 0$

Solution
The first constraint $x_1 + x_2 \leq 1$, written in a form of equation
$x_1 + x_2 = 1$
Put $x_1 = 0$, then $x_2 = 1$
Put $x_2 = 0$, then $x_1 = 1$
The coordinates are (0, 1) and (1, 0)

The first constraint $x_1 + x_2 \geq 3$, written in a form of equation

$x_1 + x_2 = 3$
Put $x_1 = 0$, then $x_2 = 3$
Put $x_2 = 0$, then $x_1 = 3$
The coordinates are (0, 3) and (3, 0)



There is no common feasible region generated by two constraints together i.e. we cannot identify even a single point satisfying the constraints. Hence there is no optimal solution.

3.4.3 Unbounded Solution

Example
Solve by graphical method
Max $Z = 3x_1 + 5x_2$
Subject to

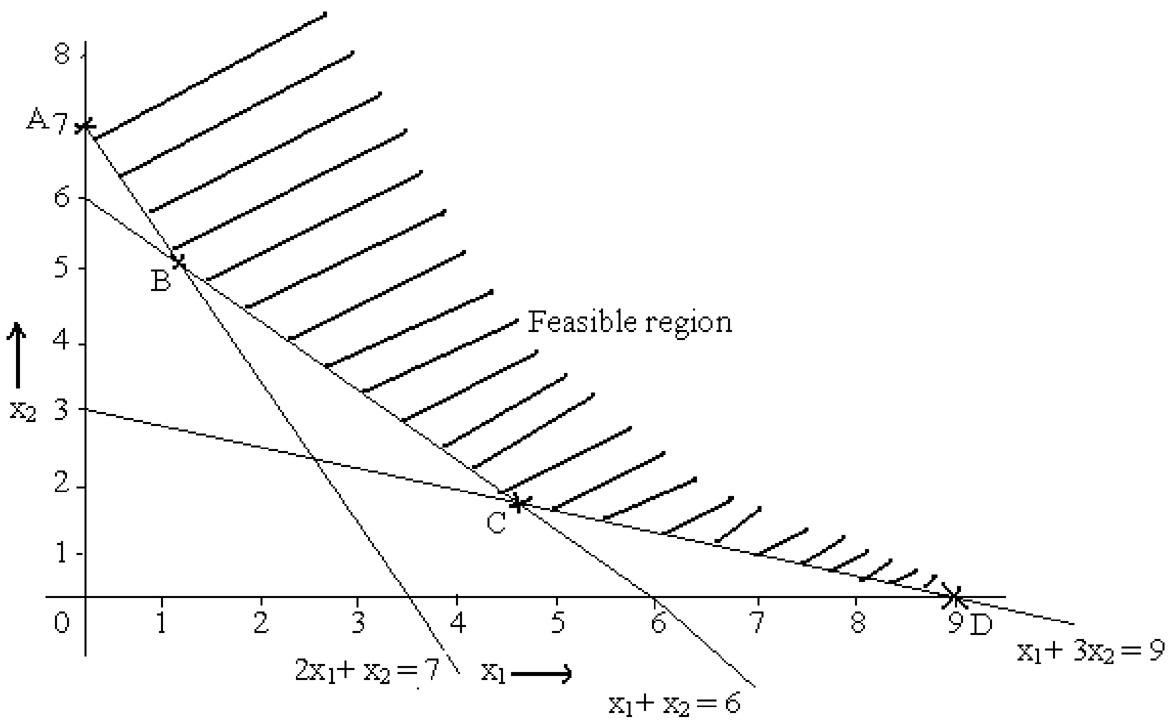
$2x_1 + x_2 \geq 7$
$x_1 + x_2 \geq 6$
$x_1 + 3x_2 \geq 9$
$x_1 \geq 0$, $x_2 \geq 0$

Solution
The first constraint $2x_1 + x_2 \geq 7$, written in a form of equation
$2x_1 + x_2 = 7$
Put $x_1 = 0$, then $x_2 = 7$
Put $x_2 = 0$, then $x_1 = 3.5$
The coordinates are (0, 7) and (3.5, 0)

The second constraint $x_1 + x_2 \geq 6$, written in a form of equation
$x_1 + x_2 = 6$
Put $x_1 = 0$, then $x_2 = 6$

Put x_2 =0, then $x_1 = 6$
The coordinates are (0, 6) and (6, 0)

The third constraint $x_1 + 3x_2 \geq 9$, written in a form of equation
$x_1 + 3x_2 = 9$
Put x_1 =0, then $x_2 = 3$
Put x_2 =0, then $x_1 = 9$
The coordinates are (0, 3) and (9, 0)



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points re

A (0, 7)

B (1, 5) (Solve the two equations $2x_1 + x_2 = 7$ and $x_1 + x_2 = 6$ to get the coordinates)

C (4.5, 1.5) (Solve the two equations $x_1 + x_2 = 6$ and $x_1 + 3x_2 = 9$ to get the coordinates)

D (9, 0)

We know that Max $Z = 3x_1 + 5x_2$

At A (0, 7)

$Z = 3(0) + 5(7) = 35$

At B (1, 5)

$Z = 3(1) + 5(5) = 28$

At C (4.5, 1.5)
$Z = 3(4.5) + 5(1.5) = 21$

At D (9, 0)
$Z = 3(9) + 5(0) = 27$
The values of objective function at corner points are 35, 28, 21 and 27. But there exists infinite number of points in the feasible region which is unbounded. The value of objective function will be more than the value of these four corner points i.e. the maximum value of the objective function occurs at a point at ∞. Hence the given problem has unbounded solution.

Introduction

General Linear Programming Problem (GLPP)

Maximize / Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots + c_nx_n$

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \ (\leq \text{ or } \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \ (\leq \text{ or } \geq)\ b_2$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \ (\leq \text{ or } \geq)\ b_m$

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions.

Introduction to Simplex Method

It was developed by G. Danztig in 1947. The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming.

The Simplex algorithm is an iterative procedure for solving LP problems in a finite number of steps. It consists of

- Having a trial basic feasible solution to constraint-equations

- Testing whether it is an optimal solution
- Improving the first trial solution by a set of rules and repeating the process till an optimal solution is obtained

Advantages

- Simple to solve the problems
- The solution of LPP of more than two variables can be obtained.

Computational Procedure of Simplex Method

Consider an example

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Write the given GLPP in the form of SLPP

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Step 2 – Present the constraints in the matrix form

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 3 – Construct the starting simplex table using the notations

		$C_j \rightarrow$	3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k

s_1 s_2	0 4 0 2	1 1	1 -1	1 0	0 1	
	$Z = C_B X_B$	Δ_j				

Step 4 – Calculation of Z and Δ_j and test the basic feasible solution for optimality by the rules given.

$Z = C_B X_B$
$= 0 * 4 + 0 * 2 = 0$

$\Delta_j = Z_j - C_j$
$= C_B X_j - C_j$
$\Delta_1 = C_B X_1 - C_j = 0 * 1 + 0 * 1 - 3 = -3$
$\Delta_2 = C_B X_2 - C_j = 0 * 1 + 0 * -1 - 2 = -2$
$\Delta_3 = C_B X_3 - C_j = 0 * 1 + 0 * 0 - 0 = 0$
$\Delta_4 = C_B X_4 - C_j = 0 * 0 + 0 * 1 - 0 = 0$

Procedure to test the basic feasible solution for optimality by the rules given

Rule 1 – If all $\Delta_j \geq 0$, the solution under the test will be **optimal**. Alternate optimal solution will exist if any non-basic Δ_j is also zero.

Rule 2 – If atleast one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.

Rule 3 – If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded**.

In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

- **Incoming vector**
  The incoming vector X_k is always selected corresponding to the most negative value of Δ_j. It is indicated by (↑).

- **Outgoing vector**
  The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6 – Mark the key element or pivot element by '□'.The element at the intersection of outgoing vector and incoming vector is the pivot element.

Cj → 3 2 0 0

Basic Variables	C_B	X_B	X_1 (X_k)	X_2	S_1	S_2	Min ratio X_B /X_k
s_1	0	4	1	1	1	0	4 / 1 = 4
s_2	0	2	[1]	-1	0	1	2 / 1 = 2 ⟶ outgoing
	Z= C_B X_B = 0		↑incoming Δ_1= -3	Δ_2= -2	Δ_3=0	Δ_4=0	

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column X_k.

Basic Variables	C_B	X_B	X_1	X_2 (X_k)	S_1	S_2	Min ratio X_B /X_k
s_1	0	2	(R_1=R_1 – R_2) 0	[2]	1	-1	2 / 2 = 1 ⟶ outgoing
x_1	3	2	1	-1	0	1	2 / -1 = -2 (neglect in case of negative)
	Z=0*2+3*2= 6		Δ_1=0	↑incoming Δ_2= -5	Δ_3=0	Δ_4=3	

Step 7 – Now repeat step 4 through step 6 until an optimal solution is obtained.

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B /X_k
x_2	2	1	(R_1=R_1 / 2) 0	1	1/2	-1/2	
x_1	3	3	(R_2=R_2 + R_1) 1	0	1/2	1/2	
	Z = 11		Δ_1=0	Δ_2=0	Δ_3=5/2	Δ_4=1/2	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 11, x_1 = 3 and x_2 = 1

Example 2

Maximize $Z = 5x_1 + 3x_2$

Subject to

$3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$
Subject to

$3x_1 + 5x_2 + s_1 = 15$
$5x_1 + 2x_2 + s_2 = 10$
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

			$C_j \rightarrow$ 5	3	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	15	3	5	1	0	15 / 3 = 5
s_2	0	10	[5]	2	0	1	10 / 5 = 2 ⟶ outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -5$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$	
s_1	0	9	$(R_1 = R_1 - 3R_2)$ 0	[19/5]	1	-3/5	9/19/5 = 45/19 ⟶
x_1	5	2	$(R_2 = R_2 / 5)$ 1	2/5	0	1/5	2/2/5 = 5
	Z = 10		$\Delta_1 = 0$	↑ $\Delta_2 = -1$	$\Delta_3 = 0$	$\Delta_4 = 1$	
x_2	3	45/19	$(R_1 = R_1 / 19/5)$ 0	1	5/19	-3/19	
x_1	5	20/19	$(R_2 = R_2 - 2/5\, R_1)$ 1	0	-2/19	5/19	
	Z = 235/19		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 5/19$	$\Delta_4 = 16/19$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 235/19, $x_1 = 20/19$ and $x_2 = 45/19$

Example 3

Maximize $Z = 5x_1 + 7x_2$
Subject to

$x_1 + x_2 \leq 4$
$3x_1 - 8x_2 \leq 24$
$10x_1 + 7x_2 \leq 35$
and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 4$
$3x_1 - 8x_2 + s_2 = 24$
$10x_1 + 7x_2 + s_3 = 35$
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	5	7	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	4	1	[1]	1	0	0	$4/1 = 4 \rightarrow$ outgoing
s_2	0	24	3	-8	0	1	0	–
s_3	0	35	10	7	0	0	1	$35 / 7 = 5$
	$Z = C_B X_B = 0$		-5	-7 ↑incoming	0	0	0	$\leftarrow \Delta_j$
x_2	7	4	1	1	1	0	0	
s_2	0	56	$(R_2 = R_2 + 8R_1)$ 11	0	8	1	0	
s_3	0	7	$(R_3 = R_3 - 7R_1)$ 3	0	-7	0	1	
	$Z = 28$		2	0	7	0	0	$\leftarrow \Delta j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 28$, $x_1 = 0$ and $x_2 = 4$

Computational Procedure of Big – M Method (Charne's Penalty Method)

Step 1 – Express the problem in the standard form.

Step 2 – Add non-negative artificial variable to the left side of each of the equations corresponding to the constraints of the type '≥' or '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).

On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by –M for maximization problems (+M for minimizing problem), where M > 0.

Step 3 – In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

2.2 Worked Examples

Example 1

Max $Z = -2x_1 - x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 – s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

		$C_j \rightarrow$	-2	-1	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	3	[3]	1	0	0	1	0	3 / 3 = 1→
a_2	-M	6	4	3	-1	0	0	1	6 / 4 = 1.5
s_2	0	4	1	2	0	1	0	0	4 / 1 = 4
	Z = -9M		↑ 2 – 7M	1 – 4M	M	0	0	0	←Δ_j
x_1	-2	1	1	1/3	0	0	X	0	1/1/3 = 3
a_2	-M	2	0	[5/3]	-1	0	X	1	6/5/3 = 1.2→
s_2	0	3	0	5/3	0	1	X	0	4/5/3 = 1.8
	Z = -2 – 2M		0	↑ $\frac{(-5M+1)}{3}$	0	0	X	0	←Δ_j
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained
Therefore the solution is Max Z = -12/5, $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

SLPP

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 3$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

		$C_j \rightarrow$	3	-1	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-M	2	[2]	1	-1	0	0	1	2 / 2 = 1→
s_2	0	3	1	3	0	1	0	0	3 / 1 = 3
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		↑ -2M-3	-M+1	M	0	0	0	←Δ_j
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	[1/2]	1	0	X	2/1/2 = 4→
s_3	0	4	0	1	0	0	1	X	-
	Z = 3		0	5/2	↑ -3/2	0	0	X	←Δ_j
x_1	3	3	1	3	0	1/2	0	X	
s_1	0	4	0	5	1	2	0	X	
s_3	0	4	0	1	0	0	1	X	
	Z = 9		0	10	0	3/2	0	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 9, $x_1 = 3$, $x_2 = 0$

Steps for Two-Phase Method

The process of eliminating artificial variables is performed in **phase-I** of the solution and **phase-II** is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as **Two-Phase Simplex Method**.

Phase I – In this phase, the simplex method is applied to a specially constructed **auxiliary linear programming problem** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 – Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2 – Construct the Auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step 3 – Solve the auxiliary problem by simplex method until either of the following three possibilities do arise

i. Max $Z^* < 0$ and atleast one artificial vector appear in the optimum basis at a positive level ($\Delta_j \geq 0$). In this case, given problem does not possess any feasible solution.

ii. Max $Z^* = 0$ and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase-II.

iii. Max $Z^* = 0$ and no one artificial vector appears in the optimum basis. In this case also proceed to phase-II.

Phase II – Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

Worked Examples

Example 1

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$
$x_2 + s_3 = 4$
$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

Auxiliary LPP
Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$
Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$
$x_1 + 3x_2 + s_2 = 2$
$x_2 + s_3 = 4$
$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-1	2	[2]	1	-1	0	0	1	1→
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
	$Z^* = -2$		↑ -2	-1	1	0	0	0	$\leftarrow \Delta_j$
x_1	0	1	1	1/2	-1/2	0	0	X	
s_2	0	1	0	5/2	1/2	1	0	X	
s_3	0	4	0	1	0	0	1	X	
	$Z^* = 0$		0	0	0	0	0	X	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		$C_j \rightarrow$	3	-1	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	[1/2]	1	0	2→
s_3	0	4	0	1	0	0	1	-
	$Z = 3$		0	5/2	↑ -3/2	0	0	$\leftarrow \Delta_j$

x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 6, $x_1 = 2$, $x_2 = 0$

Example 2

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Phase I

	$C_j \rightarrow$		0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	$\boxed{4}$	0	-1	0	0	1	$1 \rightarrow$
s_3	0	5	1	1	0	0	1	0	0	5
	$Z^* = -7$		-4	$\uparrow$ -6	1	1	0	0	0	$\leftarrow \Delta_j$

a_1	-1	1	[5/2]	0	-1	1/2	0	1	X	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	X	4
s_3	0	4	3/4	0	0	1/4	1	0	X	16/3
	Z* = -1		↑ -5/2	0	1	-1/2	0	0	X	←Δ_j
x_1	0	2/5	1	0	-2/5	1/5	0	X	X	
x_2	0	9/10	0	1	1/10	-3/10	0	X	X	
s_3	0	37/10	0	0	3/10	1/10	1	X	X	
	Z* = 0		0	0	0	0	0	X	X	←Δ_j

Since all $\Delta_j \geq 0$, Max Z* = 0 and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	C_j →		5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	5	2/5	1	0	-2/5	[1/5]	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
	Z = 46/5		0	0	-6/5	↑ -7/5	0	←Δ_j
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	[1/2]	0	1	7→
	Z = 12		7	0	↑ -4	0	0	←Δ_j
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	Z = 40		3	0	0	0	4	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 40, $x_1 = 0$, $x_2 = 5$

The Revised Simplex Method

While solving linear programming problem on a digital computer by regular simplex method, it requires storing the entire simplex table in the memory of the computer table, which may not be feasible for very large problem. But it is necessary to calculate each table during each iteration. The revised simplex method which is a modification of the original method is more economical on the computer, as it computes and stores only the relevant information needed currently for esting and / or improving the current solution. i.e. it needs only

- The net evaluation row Δ_j to determine the non-basic variable that enters the basis.
- The pivot column
- The current basis variables and their values (X_B column) to determine the minimum positive ratio and then identify the basis variable to leave the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basis matrix at any iteration.

There are two standard forms for revised simplex method

- **Standard form-I** – In this form, it is assumed that an identity matrix is obtained after introducing slack variables only.
- **Standard form-II** – If artificial variables are needed for an identity matrix, then two-phase method of ordinary simplex method is used in a slightly different way to handle artificial variables.

Steps for solving Revised Simplex Method in Standard Form-I

Solve by Revised simplex method

Max $Z = 2x_1 + x_2$

Subject to

$3x_1 + 4x_2 \leq 6$

$6x_1 + x_2 \leq 3$

and $x_1, x_2 \geq 0$

SLPP

Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2$

Subject to

$3x_1 + 4x_2 + s_1 = 6$

$6x_1 + x_2 + s_2 = 3$

and $x_1, x_2, s_1, s_2 \geq 0$

Step 1 – Express the given problem in standard form – I

- Ensure all $b_i \geq 0$
- The objective function should be of maximization
- Use of non-negative slack variables to convert inequalities to equations

The objective function is also treated as first constraint equation

$$Z - 2x_1 - x_2 + 0s_1 + 0s_2 = 0$$
$$3x_1 + 4x_2 + s_1 + 0s_2 = 6 \qquad \text{-- (1)}$$
$$6x_1 + x_2 + 0s_1 + s_2 = 3$$

and $x_1, x_2, s_1, s_2 \geq 0$

Step 2 – Construct the starting table in the revised simplex form

Express (1) in the matrix form with suitable notation

$$\begin{array}{ccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \overset{X_B}{\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}}$$

Column vector corresponding to Z is usually denoted by e_1. It is the first column of the basis matrix B_1, which is usually denoted as $B_1 = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)} \ldots \beta_n^{(1)}]$

Hence the column $\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}$ constitutes the basis matrix B_1 (whose inverse B_1^{-1} is also B_1)

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0		
s_1	0	1	0	6		
s_2	0	0	1	3		

$a_1^{(1)}$	$a_2^{(1)}$
-2	-1
3	4
6	1

Step 3 – Computation of Δ_j for $a_1^{(1)}$ and $a_2^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -2 + 0 * 3 + 0 * 6 = -2$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -1 + 0 * 4 + 0 * 1 = -1$

Step 4 – Apply the test of optimality

Both Δ_1 and Δ_2 are negative. So find the most negative value and determine the incoming vector.

Therefore most negative value is $\Delta_1 = -2$. This indicates $a_1^{(1)}$ (x_1) is incoming vector.

Step 5 – Compute the column vector X_k

$$X_k = B_1^{-1} * a_1^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$$

Stp 6 – Determine the outgoing vector. We are not supposed to calculate for Z row.

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0	-2	-
s_1	0	1	0	6	3	2
s_2	0	0	1	3	[6] ↑ incoming	1/2→outgoing

Step 7 – Determination of improved solution

Column e_1 will never change, x_1 is incoming so place it outside the rectangular boundary

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	0	0	-2
R_2	1	0	6	3
R_3	0	1	3	[6]

Make the pivot element as 1 and the respective column elements to zero.

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	1/3	1	0
R_2	1	-1/2	9/2	0
R_3	0	1/6	1/2	1

Construct the table to start with second iteration

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1		
s_1	0	1	-1/2	9/2		
x_1	0	0	1/6	1/2		

$a_4^{(1)}$	$a_2^{(1)}$
0	-1
0	4
1	1

$\Delta_4 = 1 * 0 + 0 * 0 + 1/3 * 1 = 1/3$

$\Delta_2 = 1 * -1 + 0 * 4 + 1/3 * 1 = -2/3$

Δ_2 is most negative. Therefore $a_2^{(1)}$ is incoming vector.
Compute the column vector

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} * \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}$$

Determine the outgoing vector

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1	-2/3	-
s_1	0	1	-1/2	9/2	7/2	9/7→outgoing
x_1	0	0	1/6	1/2	1/6	3
					↑ incoming	

Determination of improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2
R_1	0	1/3	1	-2/3
R_2	1	-1/2	9/2	7/2
R_3	0	1/6	1/2	1/6

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2
R_1	4/21	5/21	13/7	0
R_2	2/7	-1/7	9/7	1
R_3	-1/21	8/42	2/7	0

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	4/21	5/21	13/7		
x_2	0	2/7	-1/7	9/7		
x_1	0	-1/21	8/42	2/7		

$a_4^{(1)}$	$a_3^{(1)}$
0	0
0	1
1	0

$\Delta_4 = 1 * 0 + 4/21 * 0 + 5/21 * 1 = 5/21$
$\Delta_3 = 1 * 0 + 4/21 * 1 + 5/21 * 0 = 4/21$

Δ_4 and Δ_3 are positive. Therefore optimal solution is Max Z = 13/7, $x_1 = 2/7$, $x_2 = 9/7$