A tightly stretched string with fixed end points is initially displaced in the position $y = y_0 \sin^2 \left( \frac{m \pi}{2} \right)$ and then released from rest. Find the displacement $y$ at any distance $x$ from one end at time $t$.

**Solution**

$$y(0,t) = 0$$

The displacement $y(x,t)$ is from

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

The conditions are:

1. $y(0,t) = 0$
2. $y(l,t) = 0$
3. $\frac{\partial y}{\partial t} t=0 = 0$
4. $y(x,0) = y_0 \sin^2 \left( \frac{m \pi x}{l} \right)$

The suitable solution is

$$y(x,t) = (C_1 \cos px + C_2 \sin px) \left( C_3 \cos \omega t + C_4 \sin \omega t \right)$$

Apply (i) $y(0,t) = 0$

$$C_1 \left( C_3 \cos \omega t + C_4 \sin \omega t \right) = 0$$

$$C_1 = 0$$

Apply (ii) in (i) and (iii)

$$y(x,t) = C_2 \sin p x \left( C_3 \cos \omega t + C_4 \sin \omega t \right)$$

$$C_2 \sin p x = 0$$

$$\sin p x = 0 \Rightarrow px = n \pi \Rightarrow p = \frac{n \pi}{l}$$

(i) becomes

$$y(x,t) = \frac{C_2}{l} \sin \left( \frac{n \pi x}{l} \right) \left( C_3 \cos \left( \frac{n \pi \omega t}{l} \right) + C_4 \sin \left( \frac{n \pi \omega t}{l} \right) \right)$$
The most general solution is given by:

\[ y(x,t) = \sum_{n=1}^{\infty} \left( C_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right) + D_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi ct}{L} \right) \right) \]

First, we apply the condition at \( t = 0 \):

\[ y(0,t) = 0 \quad \Rightarrow \quad C_n \sin(0) \cos(0) + D_n \sin(0) \sin(0) = 0 \]
\[ C_n + D_n = 0 \]

Next, we apply the condition at \( x = \frac{L}{2} \):

\[ y \left( \frac{L}{2}, t \right) = 0 \quad \Rightarrow \quad C_n \sin \left( \frac{n\pi \frac{L}{2}}{L} \right) \cos \left( \frac{n\pi ct}{L} \right) = 0 \]
\[ C_n \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{n\pi ct}{L} \right) = 0 \]

Since \( \sin \left( \frac{n\pi}{2} \right) \) alternates between 0 and 1, we need:

\[ \cos \left( \frac{n\pi ct}{L} \right) = 0 \]

This occurs when:

\[ n\pi \frac{ct}{L} = \frac{\pi}{2} \quad \Rightarrow \quad n = \frac{2}{2} \]

Thus, all terms with even \( n \) vanish, and we have:

\[ y(x,t) = \sum_{n=1}^{\infty} \left( C_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right) \right) \]

On comparing:

\[ 2 \frac{y_0}{L} \sin \left( \frac{n\pi x}{L} \right) - \frac{y_0}{4} \sin \left( \frac{3n\pi x}{L} \right) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{L} \right) \]

Comparing coefficients:

\[ C_n = \frac{2 \frac{y_0}{L} - \frac{y_0}{4} \sin \left( \frac{3n\pi x}{L} \right)}{2} \]

For \( n = 1 \):

\[ C_1 = \frac{2 \frac{y_0}{L} - \frac{y_0}{4} \sin \left( \frac{3\pi x}{L} \right)}{2} \]

For \( n = 2 \):

\[ C_2 = 0 \]

For \( n = 3 \):

\[ C_3 = -\frac{3y_0}{4} \]

For \( n > 3 \):

\[ C_n = 0 \]

Thus, the solution is:

\[ y(x,t) = \sin \left( \frac{\pi x}{L} \right) \cos \left( \frac{\pi ct}{L} \right) - \frac{3y_0}{4} \sin \left( \frac{3\pi x}{L} \right) \cos \left( \frac{3\pi ct}{L} \right) \]
A string of length $2l$ is fastened at both ends. The mid pt. of the string is taken to a height $b$ and then released from rest in that position. Show that the displacement is 

$$y(x, t) = b \sin \left( \frac{n \pi x}{2l} \right) \sin \left( \frac{n \pi ct}{2l} \right)$$

A tightly stretched string of length $l$ has its ends fastened at $x = 0$ and $x = l$. The mid point on the string is then taken to a height $h$ and then released from rest in that position. Obtain an expression for the displacement of the string at any subsequent time.

**Solution:** The equation to be solved is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions:

1. $y(0, t) = 0$
2. $y(l, t) = 0$
3. $y_x(x, 0) = 0$
4. $y(0, t) = 0$

Eq. of AD is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \Rightarrow \frac{x - 0}{1 - 0} = \frac{y - 0}{h - 0}$$

$$y = h$$

Eq. of DB is

$$\frac{x - \frac{l}{2}}{1 - \frac{l}{2}} = \frac{y - h}{0 - h} \Rightarrow \frac{x - \frac{l}{2}}{1 - \frac{l}{2}} = -1$$

$$y - h = -2h (x - \frac{l}{2}) \Rightarrow y = h - 2hx + h$$

$$y(0) = \frac{2h}{1} \left[ 1 - \frac{x}{l} \right] \Rightarrow 0 < x < \frac{l}{2}$$

$$y(l) = \frac{2h}{1} \left[ 1 - \frac{2l}{l} \right] = 0 \Rightarrow \frac{l}{2} < x < l$$
Applications Of Partial Differential Equations

Applying continuity in (5)

\[ y(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \]

To find \( A_n \), expand the value in a half-range sine series

\[ A_n = \frac{2}{L} \int_{0}^{\frac{L}{2}} f(x) \sin \frac{n\pi x}{L} \, dx \]

\[ A_n = \frac{2}{L} \left[ \frac{1}{2} \cos \frac{n\pi x}{L} + \left( \frac{L}{n\pi} \right)^2 \sin \frac{n\pi x}{L} \right]_{0}^{\frac{L}{2}} \]

\[ C_n = \frac{2}{L} \left[ \int_{0}^{\frac{L}{2}} \sin \frac{n\pi x}{L} \, dx + \int_{\frac{L}{2}}^{L} (1-x) \sin \frac{n\pi x}{L} \, dx \right] \]
Applications Of Partial Differential Equations

\[ \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \]

\[ y(x,0) = f(x) \]
\[ \frac{\partial y}{\partial t} (x,0) = g(x) \]
\[ y(0,t) = h(t) \]
\[ y(L,t) = 0 \]

**Solution:**

1. The Wave Equation is
2. \[ \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \]

Eq. of line

\[ \frac{y_y-y_1}{y_b-y_1} = \frac{a-x}{b-x} \]
\[ \frac{b-a}{b-a} = \frac{y_b-y_1}{y_b-y_1} \]

**Eq. of BB**

\[ y_b-y_1 = a-x \]
\[ y_b-y_1 = b-x \]

\[ y(x,0) = \int_{0}^{L} \frac{b_x}{x} \left( \frac{b(a-x)}{a} \right) \]

\[ y(x,t) = \sum_{n=1}^{\infty} \left( c_n \cos \frac{\pi x}{L} + d_n \sin \frac{\pi x}{L} \right) \]

\[ \frac{b_1}{x} \left( \frac{b(a-x)}{a} \right) \]

To find \( c_n \), expand the given function in a half-range Fourier sine series in the interval (0, L) \((L \leq a)\)

\[ b_n = \frac{2}{aL} \int_{0}^{a} f(x) \sin \frac{\pi x}{a} \]
Applications Of Partial Differential Equations

Problems on Vibrating String with non-zero initial velocity.

The boundary and initial conditions of the deflection $y(x,t)$ are

1. $y(x,0) = 0$
2. $y(1,t) = 0$
3. $y(0,t) = 0$
4. $\frac{\partial^2 y}{\partial t^2} (0,t) = f(t)$

The suitable solution is

$y(x,t) = (c_1 \cos px + c_2 \sin px) (c_3 \cos pt + c_4 \sin pt)$

Apply $t = 0$. We set $c_1 = 0$

$\rightarrow c_2 = 0$

The most general solution is

$y(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} \left( c_n \sin \frac{nx}{L} \right) \left( b_n \sin \frac{n\pi t}{L} \right)$

Apply $t = 0$ Conv.

$\frac{dy}{dx} (0,t) = \frac{1}{2} \sum_{n=1}^{\infty} \left( c_n \sin \frac{nx}{L} \right) b_n \sin \frac{n\pi t}{L} = 0$

where

$B_n = \frac{1}{L} \int_0^L \sin \frac{nx}{L} \sin \frac{n\pi t}{L} dx$

$B_n = b_n = \frac{2}{L} \int_0^L \sin \frac{nx}{L} \sin \frac{n\pi t}{L} dx$

$C_n = \frac{1}{|B_n|} \frac{1}{c_n}$

A tightly stretched string with fixed end $x = 0$ and $x = L$ is initially at rest in its equilibrium position. If it is set vibrating, string giving each point a velocity $\pm (1-x)$ show that the displacement is

$y(x,t) = \frac{8aL^3}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)x}{L} \sin \frac{(2n-1)(2n-1)t}{L}$
Applications Of Partial Differential Equations

Boundary & Initial Cond.
1. \( y(0,t) = 0 \)
2. \( y(l,t) = 0 \)
3. \( y(x,0) = f(x) \quad 0 < x < l \)
4. \( \frac{\partial y}{\partial t} \bigg|_{t=0} = \lambda^2 (l-x) \quad 0 < x < l \)

The suitable sol. is

\[ y(x,t) = C_1 \cos(n \pi x) + C_2 \sin(n \pi x) \]

Apply Cond. 1 in I

\[ y(0,t) = C_2 \sin(n \pi x) = 0 \]

Substitute \( C_2 = 0 \) in I

\[ y(x,t) = C_1 \cos(n \pi x) \]

Apply Cond. 2 in II

\[ y(1,t) = C_1 \cos(n \pi) = 0 \]

Here \( C_1 \cos(n \pi) = 0 \).

Suppose \( C_1 = 0 \) already we have \( y(x,t) = 0 \).

Now, \( y(x,t) \) in eq. 2

\[ y(x,t) = C_2 \sin(n \pi x) \]

Apply Cond. 3 in III

\[ y(x,0) = C_3 \sin(n \pi x) = 0 \]

Simplify \( C_3 = 0 \).

Therefore \( y(x,t) = 0 \).
\[ C_0 \cos \frac{m\pi x}{L} \leq 0 \]
\[ \sin \frac{n\pi x}{L} \neq 0 \quad (it \text{ is defined for odd } n) \]
\[ C_n = 0. \]
\[ C_3 = 0. \]

Substitute \( C_3 = 0 \) in \( \text{eq 8} \)

\[ y(x,t) = C_0 \cos \frac{m\pi x}{L} \sin \frac{n\pi \eta}{2} \]

The most general form:

\[ y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sin \frac{n\pi \eta}{2} - \frac{2}{n} \]

Before applying boundary conditions:

\[ \frac{\partial y}{\partial t} (x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \left( \frac{n\pi}{L} \right) \frac{\sin \frac{n\pi \eta}{2}}{2} - \frac{2}{n} \]

Apply boundary conditions:

\[ \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi}{L} C_n \sin \frac{n\pi x}{L} \sin \frac{n\pi \eta}{2} = \frac{2}{n} \]

\[ B_n = C_n \eta \sin \frac{n\pi \eta}{2} \]

To find \( B_n \):

Endpoints \( x = (L-x) \) at both ends.

Using sine series,

\[ \lambda_{1}(L-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \]

\[ B_n = \frac{2}{L} \int_{0}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi \eta}{2} \, dx \]
Applications Of Partial Differential Equations

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} \right) = \frac{\partial^2 u}{\partial x^2} \]

\[ \frac{\sin \frac{n\pi x}{L}}{\frac{n\pi}{L}} - \alpha \left( \frac{n\pi}{L} \right)^2 \frac{\sin \frac{n\pi c^2 t}{L}}{\frac{n\pi}{L}} \]

\[ = \frac{\partial^2 u}{\partial x^2} \left[ (-n^2 \left( \frac{n\pi}{L} \right)^2 \cos \left( \frac{n\pi x}{L} \right) - 1 - \alpha \left( \frac{n\pi}{L} \right)^2 \right] \]

\[ = \frac{\partial^2 u}{\partial x^2} \left[ \left( \frac{n\pi}{L} \right)^2 \sin \left( \frac{n\pi x}{L} \right) + 2 \left( \frac{n\pi}{L} \right)^2 \right] \]

\[ = \frac{\partial^2 u}{\partial x^2} \left[ \frac{4}{\frac{n\pi}{L}} \left( 1 - (-1)^n \right) \right] \]

\[ c_n = \frac{4}{n^3 \pi^3} \left( 1 - (-1)^n \right) \]

\[ y_{even} = \sum_{n=0}^{\infty} \frac{8n^2}{n^2 \pi^2} \left( 1 - (-1)^n \right) \sin \left( \frac{n\pi x}{L} \right) \]

\[ y_{odd} = \sum_{n=1}^{\infty} \frac{8n^2}{n^2 \pi^2} \left( 1 - (-1)^n \right) \sin \left( \frac{n\pi x}{L} \right) \]

A tightly stretched string of length \( L \) is initially at rest in its equilibrium position and each of its Pts is given the velocity \( u_0 \). Find the displacement \( \phi \) of the wave.

The wave Eq is \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \)

The boundary conditions are \( y(0,t) = 0 \) \( y(L,t) = 0 \)

\( \phi \) of free end
Applications Of Partial Differential Equations

\[
\begin{align*}
\frac{3v_0}{4} \sin \left( \frac{3\pi n}{2} \right) - \frac{v_0}{4} \sin \left( \frac{3\pi n}{2} \right) &= \frac{3v_0}{4} \sin \left( \frac{3\pi n}{2} \right) - \frac{v_0}{4} \sin \left( \frac{3\pi n}{2} \right) \\
&= B_1 \left( \frac{3\pi n}{2} \right) \sin \left( \frac{3\pi n}{2} \right) + B_2 \sin \left( \frac{3\pi n}{2} \right) + B_3 \sin \left( \frac{3\pi n}{2} \right) \\
B_1 &= \frac{3v_0}{4} \quad B_2 = -\frac{v_0}{4} \\
B_3 &= \frac{v_0}{4} \left( \frac{\pi n}{2} \right) \\
\frac{3v_0}{4} \sin \left( \frac{3\pi n}{2} \right) &= B_3 \sin \left( \frac{3\pi n}{2} \right) \\
B_3 &= C_3 \left( \frac{3\pi n}{2} \right) = \frac{v_0}{4} \\
C_3 &= -\frac{v_0}{12\pi n} \\
\text{Limit} &= \frac{3v_0}{4} \sin \left( \frac{3\pi n}{2} \right) \sin \left( \frac{3\pi n}{2} \right) = \frac{v_0}{2} \sin \left( \frac{3\pi n}{2} \right) \sin \left( \frac{3\pi n}{2} \right)
\end{align*}
\]

8. A string of length \( L \) is initially at rest in the equilibrium position and motion is started by giving each of the bits a velocity \( V \). Find the displacement \( V \) for \( 0 \leq x \leq \frac{L}{2} \).

\( V = \frac{3}{4} c \sin \left( \frac{1}{2} \right) \left( L - x \right) \)

2011. The wave equation.

boundary cond.

\[ \psi \left( x, 0 \right) = 0 \quad \psi \left( 0, t \right) = 0 \quad 0 \leq x \leq \frac{L}{2} \]

\[ \left( \frac{\partial^2 \psi}{\partial t^2} \right) \left( x, 0 \right) = 0 \quad 0 \leq x \leq \frac{L}{2} \]

\[ \psi \left( x, t \right) = \left( C_1 \cos \pi x + C_2 \sin \pi x \right) \left( C_3 \cos \pi \left( \frac{L - x}{L} \right) + C_4 \sin \pi \left( \frac{L - x}{L} \right) \right) \]
Applications Of Partial Differential Equations

Apply Cond (14)

\[ \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) = \begin{cases} \sin \left( \frac{\pi x}{L} \right), & 0 < x < L \\ 0, & x = 0, L \end{cases} \]

So \( B_n = C_n \left( \frac{\pi n}{L} \right) \)

No find \( B_n \),

\[ b_n = \frac{1}{L} \int_0^L \sin \left( \frac{n\pi x}{L} \right) dx \]

\[ b_n = \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) \]

\[ C_n = \frac{1}{L} \int_0^L C \cos \left( \frac{n\pi x}{L} \right) dx \]

\[ = \frac{2C}{n\pi} \left[ \frac{\cos \left( \frac{n\pi}{2} \right)}{n\pi} \right] \]

\[ = \frac{2C}{n\pi} \left[ \frac{1}{n\pi} \right] \sin \left( \frac{n\pi}{2} \right) \]

\[ = \frac{C}{n\pi} \sin \left( \frac{n\pi}{2} \right) \]

\[ B_n = \frac{C}{n\pi} \sin \left( \frac{n\pi}{2} \right) \]

\[ u(x,t) = \frac{L^2}{\alpha} \sum_{n=1}^{\infty} \frac{C}{n\pi} \sin \left( \frac{n\pi}{2} \right) \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi t}{L} \right) \]

\[ u(x,t) = \sum_{n=1}^{\infty} \frac{C}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi t}{L} \right) \]
Applications Of Partial Differential Equations

One dimensional Heat eqn.
The One dimensional heat eqn is

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \]

\[ \alpha^2 = \frac{k}{\rho c} \rightarrow \text{Diffusivity of the Material} \]

\[ k \rightarrow \text{Thermal Conductivity} \]

\[ \rho \rightarrow \text{Density} \]

\[ c \rightarrow \text{Specific heat} \]

Temperature for \( x \) is \( u(\text{unit}) \)

Note: \( \alpha^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \)

\[ \frac{\partial^2}{\partial x^2} v(x) + \frac{\partial^2}{\partial y^2} v(y) + \frac{\partial^2}{\partial z^2} v(z) = 0 \]

The One dimensional heat eqn is \( \text{parabolic} \)

Write all possible soln. of one dimensional heat eqn.

1. \( u(x,t) = \left[ A e^{\alpha x} + B e^{-\alpha x} \right] e^{-\alpha^2 \beta t} \)
2. \( u(x,t) = \left[ A e^{\alpha x} + B e^{-\alpha x} \right] e^{-\alpha^2 \beta t} \)

A rod 30 cm long has its ends A and B kept at 20°C and 80°C respectively. Until steady state conditions prevail, the temp at each end is then suddenly reduced to 0°C and kept so on. Find the resulting
Applications Of Partial Differential Equations

The temperature \( U(x,t) \) is given by:

\[
\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = \frac{\partial U}{\partial t}
\]

The boundary conditions are:

1. \( U(0,t) = 0 \)
2. \( U(30,t) = 0 \)
3. \( U(90,0) = 2a + b = 2x + 20 \)

The solution:

\[
U(x,t) = A \cos \left( \frac{2\pi x}{90} \right) e^{-\frac{\pi^2 t}{900}}
\]

Apply (i): Put \( x = 0 \)

\[
0 = A e^{-2\pi^2 t}
\]

\[
A = 0
\]

Apply (ii): Put \( x = 30 \)

\[
0 = 2A \sin(20p) e^{-\frac{2\pi^2 t}{900}}
\]

\[
B \sin(20p) = 0
\]

\[
\sin(20p) = 0 \Rightarrow 20p = n\pi \Rightarrow p \Rightarrow n\pi
\]

\[
\Rightarrow 0 \text{ becomes}
\]

\[
U(x,0) = B \sin \left( \frac{n\pi x}{90} \right) e^{-\frac{\pi^2 t}{900}}
\]

The most general solution is:

\[
U(x,t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{90} \right) e^{-\frac{n^2 \pi^2 t}{900}}
\]

Apply (iii): Put \( t = 0 \)

\[
2x + 20 = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{90} \right)
\]
Applications Of Partial Differential Equations

\[
C_n = \frac{2}{30} \int_0^{20} \left( 2n + 20 \right) \sin \left( \frac{\pi n x}{30} \right) \, dx
\]

\[
= \frac{1}{15} \left[ \left( 2n + 20 \right) \left( -\cos \left( \frac{n \pi x}{30} \right) \right) \right]_{x=0}^{x=20} + 2 \left( \frac{\sin \left( \frac{n \pi x}{30} \right)}{n \pi} \right)_{x=0}^{x=20}
\]

\[
= \frac{1}{15} \left[ \frac{-6400}{n \pi} + \frac{600}{n \pi} \left( \frac{1}{n} \right) \right]
\]

\[
= \frac{600}{15 \pi} \left[ -4 \left( -1 \right)^n + 1 \right]
\]

\[
= \frac{40}{n \pi} \left[ -4 \left( -1 \right)^n + 1 \right]
\]

\[
c(x, y) = \sum_{n=1}^\infty \frac{40}{n \pi} \left[ -4 \left( -1 \right)^n + 1 \right] \sin \left( \frac{n \pi x}{30} \right)
\]

\[
e^{-\frac{\pi^2 y^2}{900}}
\]
A rod of length 1 has its ends A and B kept at 0°C and 100°C respectively. Until steady state conditions prevail. If the temp. at B is reduced suddenly to 0°C and kept so while that at A is maintained.

Find the temperature at A (t = 0).

Sol.

The temp. at A (t = 0) is from

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \]

The cond. are:

1. \( u(0,t) = 0 \)
2. \( u(l,t) = 0 \)
3. \( u(x,0) = 100 \times e^{-\frac{x}{l}} \)

The suitable sol. is

\[ u(x,t) = (A \cos \frac{\pi x}{l} + B \sin \frac{\pi x}{l}) e^{-\frac{\pi^2}{l^2} t} \]

Apply 3

\[ A(ce^{-\frac{\pi^2}{l^2} t}) = 0 \]

Apply 2

\[ u(l,t) = A \cos \frac{\pi l}{l} + B \sin \frac{\pi l}{l} (ce^{-\frac{\pi^2}{l^2} t}) = 0 \]

\[ \sin \frac{\pi l}{l} = 0 \]

The general sol. is

\[ u(x,t) = \sum_{n=1}^{\infty} \left( C_n \cos \frac{\pi nx}{l} + D_n \sin \frac{\pi nx}{l} \right) e^{-\frac{n^2 \pi^2}{l^2} t} \]

Apply 3

\[ u(l,t) = \sum_{n=1}^{\infty} \left( C_n \cos \frac{\pi nl}{l} + D_n \sin \frac{\pi nl}{l} \right) e^{-\frac{n^2 \pi^2}{l^2} t} = 0 \]

\[ C_n = \frac{1}{l} \int_0^l u(x,0) \cos \frac{\pi nx}{l} \, dx \]

\[ = \frac{200}{l^2} \left[ l - \cos \left( \frac{\pi n}{l} \right) \left( \frac{1}{n^2} \right) + \frac{l^2}{n^2 \pi^2} \sin \left( \frac{\pi n}{l} \right) \right] \]
Applications Of Partial Differential Equations

The ends A and B of a rod 40 cm long have their temp. kept at 80°C and 80°C, respectively. Until steady state is reached, the temp. at the end B is then suddenly reduced to 40°C and kept so. Write the expression for the temp. distribution U(x,t) in meters.

A

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>40 cm</td>
<td>B</td>
</tr>
<tr>
<td>0</td>
<td>40</td>
</tr>
</tbody>
</table>

\[ A(x,t) = \begin{cases} 
40 & \text{for } x = 0 \\
40 & \text{for } x = 40 \\
\frac{1}{2} (a x + b) & \text{otherwise} 
\end{cases} \]

\[ a = 80 - 40, \quad b = 40 \]

The temp. \( U(x,t) \) is from \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0 \)

The Cond. are:

1. \( u(x,0) = 40 \)
2. \( u(40,t) = 40 \)
3. \( u(0,t) = a x + b - 20 \)

The suitable soln.

\[ U(x,t) = (A \cos \omega t + B \sin \omega t) e^{-\alpha x} + U(x) \]

\[ U(x) = a x + b - 40 \]

Apply (2)

\[ (A \cos \omega t + B \sin \omega t) e^{-\alpha x} + U(x) \]
Applications Of Partial Differential Equations

Apply (2)

\[ H_0 = H_0 + \left( B \sin 40P \right) e^{-x^2 t} \]

\[ (B \sin 40P) e^{-x^2 t} = 0 \]

\[ \sin 40P = B \sin 40P \quad P = \frac{n\pi}{40} \]

\[ H(x,t) = A + B \sin \left( \frac{n\pi x}{40} \right) e^{-x^2 t} \frac{\alpha^2}{1600} \]

Most Gen. Sol. (a) C & t

\[ H(x,t) = A + \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{40} \right) e^{-x^2 t} \frac{\alpha^2}{1600} \]

Apply (3) Put t=0

\[ \sin \left( \frac{n\pi x}{40} \right) = \sin \left( \frac{n\pi x}{40} \right) \]

\[ A = \frac{\pi}{40} \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{40} \right) e^{-x^2 t} \frac{\alpha^2}{1600} \]

\[ C_n = \frac{4}{40} \int_0^{40} H(x,t) \sin \left( \frac{n\pi x}{40} \right) dx \]

\[ = \frac{1}{40} \left[ \frac{\pi}{40} \cos \left( \frac{n\pi x}{40} \right) H_0 \right]_{0}^{40} + \frac{\pi}{40} \sin \left( \frac{n\pi x}{40} \right) \left[ \frac{1600}{n^2 \pi^2} \right]_{0}^{40} \]

\[ = \frac{1}{80} \left[ -\frac{1600 \pi}{n^2 \pi^2} \cos \left( \frac{n\pi x}{40} \right) \right]_{0}^{40} - \frac{80 \pi}{n^2 \pi^2} \sin \left( \frac{n\pi x}{40} \right) \]

\[ u(x,t) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{40} \right) e^{-x^2 t} \frac{\alpha^2}{1600} \]
A metal bar 10 cm long with initial
end temperatures 20°C and 40°C. It is kept at 30°C and 40°C. Until steady state condi-
tions prevail, the temp. at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the temperature at any point at the bar at any time.

Problem: The temp. at point (t, x) is from

\[ u(x, t) = \frac{\partial^2 u}{\partial x^2} \]

The steady state:
1. \( u(t, 0) = 20°C \)
2. \( u(0, t) = 10°C \)
3. \( u(x, 0) = 20°C + 20°C \)

The initial condition:

At \( t = 0 \),

\[ u(x, 0) = \frac{\partial u}{\partial t} \]

Apply the initial condition at \( t = 0 \):

\[ 20°C = A e^{-a x} \]

Apply the condition at \( x = 10 \),

\[ 40°C = B e^{-b x} \]

The solution is:

\[ u(x, t) = \sum_{n=1}^{\infty} \left( B_n \sin \left( \frac{n\pi x}{10} \right) e^{-\alpha n^2 t} \right) \]

Most simplest:

\[ u(x, t) = \frac{20°C}{\pi} \sin \left( \frac{n\pi x}{10} \right) e^{-\alpha n^2 t} \]

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Applying the boundary conditions:

\[ 2x + 20 = \frac{2}{\pi^2} c_n \sin \left( \frac{n \pi}{10} \right) - 4x + 50 \]

\[ (5x - 30) = \frac{2}{\pi^2} c_n \sin \left( \frac{n \pi}{10} \right) \]

\[ c_n = \frac{2}{\pi^2} \int_0^{10} \left( (5x - 30) \sin \left( \frac{n \pi x}{10} \right) \right) dx \]

\[ = \frac{1}{5} \left[ -\frac{(6x-30) \sin(n\pi x/10)}{n\pi} \right]_0^{10} = \left[ \frac{300 \sin(n\pi)}{n\pi} \right] \]

\[ = \frac{1}{5} \left[ \frac{300 (e^{-n\pi} - e^{n\pi})}{n\pi} \right] \]

\[ = \frac{600}{n\pi} \left( e^{n\pi} - e^{-n\pi} \right) \]

\[ \Rightarrow n = 2, 4, 6, \ldots \]

\[ n = 1, 3, 5, \ldots \]

Equation (3) becomes:

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{120}{n\pi} \frac{\sin}{} \left( \frac{n\pi x}{10} \right) e^{\frac{2n^2\pi^2t}{100}} \]

\[ = \frac{120}{n\pi} \sin \left( \frac{n\pi x}{10} \right) e^{\frac{2n^2\pi^2t}{100}} \]

Two-dimensional Heat Eqn.

The two-dimensional heat equation is:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

The temperature function is \( u(x, y) \).

In \( A = 1 \) \( B = 0 \) \( C = 0 \)

\[ B^2 - 4AC = 0 - 4(1)(0) = -4 < 0 \]

Two-dimensional heat equation is elliptic.
Applications Of Partial Differential Equations

Write down all possible solutions of the two dimensional heat equation.

1. \( u(x,y) = (A \cos \pi x + B \sin \pi x) (C e^{Dy}) \)
2. \( u(x,y) = (A e^{\rho x} + B e^{-\rho x}) (C \cos \pi y + D \sin \pi y) \)
3. \( u(x,y) = (A x + B)(x y + D) \)

A square plate is bounded by the lines \( x=0 \), \( y=0 \), \( x=a \), and \( y=a \). The line \( x=0 \) and \( y=0 \) are kept at a constant temperature \( 0^\circ \). The sides \( x=a \) and \( y=a \) are kept at a constant temperature of \( 3\% \) by \( u(0,y)=100 \) for \( 0 < y < a \). Find \( u(x,y) \).

The temperature \( u(x,y) \) is from

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0
\end{align*}
\]

The conditions are

1. \( u(0,0)=0 \)
2. \( u(a,0)=0 \)
3. \( u(0,y)=0 \)
4. \( u(a,y)=100 \)

The suitable solution is

\( u(x,y) = (A e^{\rho x} + B e^{-\rho x}) (C \cos \pi y + D \sin \pi y) ~ \text{(3)} \)

Apply (1)

\( 0 = (A e^{\rho x} + B e^{-\rho x}) (C e^{Dy}) \)

\( \Rightarrow C = 0 \)

Apply (3) Sub \( C = 0 \) in (1)

\( u(x,y) = (A e^{\rho x} + B e^{-\rho x}) (D \sin \pi y) ~ \text{(4)} \)

Apply (3) in (1)

\( u(x,y) = 0 \)
Applications Of Partial Differential Equations

\[ D \sin \rho a = 0 \quad \rho = \frac{mk}{a} \]

\[ \sin \rho a = 0 = \sin \frac{mk}{a} \]

\[ \rho = m\% \]

Substitute \( \rho = m\% \) in \( \sum \)

\[ u(m,y) = \left( A e^{\frac{my}{a}} + B e^{-\frac{my}{a}} \right) \left( D \sin \frac{ny}{a} \right) \]

Apply (m)

\[ u(0,y) = (A+B) \cdot \left( D \sin \frac{ny}{a} \right) = 0 \]

\[ B = -A \]

\[ u \left( 0, y \right) = 0 \]

The most general form is

\[ u(m,y) = \sum \frac{2}{n+1} D_{n} \left( e^{\frac{my}{a}} - e^{-\frac{my}{a}} \right) \sin \left( \frac{n\pi y}{a} \right) \]

\[ u(m,y) = \sum \frac{2}{n+1} \int_{0}^{a} D_{n} \left( e^{\frac{my}{a}} - e^{-\frac{my}{a}} \right) \sin \left( \frac{n\pi y}{a} \right) dy \]

\[ D_{n} \left( e^{\frac{my}{a}} - e^{-\frac{my}{a}} \right) = \frac{\sin \left( \frac{n\pi y}{a} \right)}{\sin \left( \frac{n\pi y}{a} \right)} \]

\[ = \frac{200}{a} \int_{0}^{a} \sin \left( \frac{n\pi y}{a} \right) dy \]

\[ = \frac{200}{a} \left[ \cos \left( \frac{n\pi y}{a} \right) \right]_{0}^{a} \]

\[ = \frac{200}{a} \left[ -\cos \left( \frac{n\pi y}{a} \right) \right] \frac{a}{\frac{a}{2}} \]

\[ = \frac{200}{a} \left[ -a + \frac{a}{n \pi} \right] \]

\[ = \frac{200}{a} \left[ -a + \left( \frac{-1}{n \pi} \right) \right] \]

\[ n = 0, 1, 2, \ldots \]

\[ D_{n} \left( e^{\frac{my}{a}} - e^{-\frac{my}{a}} \right) = \frac{200}{a} \left[ -a + \left( \frac{-1}{n \pi} \right) \right] \]
Applications Of Partial Differential Equations

\[ D_n = \frac{400}{m} \left( e^{m} - e^{-m} \right) \]

\[ U(\eta, \mu) = \sum_{n=1,3,5} \frac{400}{m} \left( e^{\frac{m\eta}{a}} - e^{-\frac{m\eta}{a}} \right) \sin \left( \frac{m\mu}{a} \right) \]

Finite Plate with Value given in y-direction.

1. \( U(n, 0) = 0 \)
2. \( U(n, L) = 0 \)
3. \( U(0, y) = 0 \)
4. \( U(L, y) = f(y) \)

The Suitable Eqn.

\[ U(n, y) = (Ae^{ap} + Be^{-ap}) \left( C \cos p y + D \sin p y \right) \]

\[ x-direction \]

1. \( U(0, y) = 0 \)
2. \( u(L, y) = 0 \)
3. \( u(x, 0) = 0 \)
4. \( u(x, L) = f(x) \)

The Suitable Eqn.

\[ U(n, y) = (Ae^{ap} + Be^{-ap}) \left( C \cos p x + D \sin p x \right) \]

A sq. plate is bounded by the lines \( x = 0, y = 0, n = 20 \) and \( y = 20 \). Its faces are insulated, the temp. along the upper horizontal edge is \( 3 \) by \( U(n, 20) = 3 (20 - x) = 20 x - x^2 \) while the lower edges are kept at \( 6 \). Find the steady state temp. distribution in the plate.
The temp. $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The initial conditions are

1. $u(0,y) = 0$
2. $u(x,y_0) = 0$
3. $u(x,0) = 0$
4. $u(1,y) = 20x - x^2$

The suitable solution is

$$u(x,y) = [A \cos px + B \sin px] [C e^{py} + D e^{-py}]$$

Apply (1) $0 = A (C e^{py} + D e^{-py}) \Rightarrow A = 0$

Apply (2) $0 = B \sin (2py) (C e^{py} + D e^{-py})$

Apply (3)

$$B \sin (2py) = 20p = \pi \Rightarrow p = \frac{\pi}{20}$$

Apply (4)

$$D = 0$$

Become

$$u(x,y) = B \sin px \left[ C e^{py} - e^{-py} \right]$$

Most general

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{2\pi nx}{20} \right) \left[ e^{\frac{\pi ny}{20}} - e^{-\frac{\pi ny}{20}} \right]$$

Apply (4)

$$20x - x^2 = \sum_{n=1}^{\infty} C_n \sin \left( \frac{2\pi nx}{20} \right) \left[ e^{\frac{\pi ny}{20}} - e^{-\frac{\pi ny}{20}} \right]$$

$$C_n \left[ e^{\frac{\pi ny}{20}} - e^{-\frac{\pi ny}{20}} \right] = \frac{1}{20} \int_{0}^{10} (20x - x^2) \sin \left( \frac{2\pi nx}{20} \right) dx$$

$$= \frac{1}{20} \left[ \frac{400}{n^2 \pi^2} \left( 20 - 2n \right) \sin \left( \frac{2\pi nx}{20} \right) \right]_{0}^{10}$$

$$= \frac{16000}{n^2 \pi^2} \cos \left( \frac{2\pi ny}{20} \right)$$
Applications Of Partial Differential Equations

\[ c = \frac{1}{10} \left( \frac{-16000}{n^3} \right) (-1)^n + \frac{16000}{n^3} \left( \frac{1}{n^3} \right) \]

\[ = \frac{1600}{n^3} \left( -(-1)^n + 1 \right) \]

\[ = 0 \quad \text{no. 2,4,6} \]

\[ \frac{3200}{n^3} \quad n = 1,3,5 \quad \text{no. 2,4,6} \]

\[ C_n = \frac{3200}{n^3 (\sinh n\pi)} \quad \text{no. 2,4,6} \]

\[ u(x,y) = \sum_{n=1}^{\infty} \frac{3200}{n^3 (\sinh n\pi)} \sin\left( \frac{n\pi x}{b} \right) \frac{\sinh n\pi}{\sinh n\pi} \quad \text{no. 2,4,6} \]

A rectangular plate with insulated surfaces is 20cm wide and 30cm long. Considered to be infinite in length, that it may be considered infinite in length.

If the temp. at the short edge is \( T_0 \) and the two long edges are kept at

\[ u = \frac{90}{10(200y)} \quad 0 \leq y \leq 10. \]

Find the steady state temp. distribution in the

Plate.

The temp. \( u(x,y) \)

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

The boundary cases:

1. \( u(x,0) = 0 \)
2. \( u(x,10) = 0 \)
3. \( u(0,y) = 0 \)
4. \( u(20,y) = u = 10(20-y) \quad 0 \leq y \leq 20. \)
Applications Of Partial Differential Equations

The solution is:

\[ U(x,y) = \left( A e^{x} + B e^{-x} \right) \left( C \cos y + D \sin y \right) \]

Apply (i)

\[ \frac{\partial}{\partial x} \left( A e^{x} + B e^{-x} \right) = 0 \]

Apply (ii)

\[ \frac{\partial}{\partial y} \left( C \cos y + D \sin y \right) = 0 \]

Apply (iii)

\[ \frac{\partial}{\partial x} \left( A e^{x} + B e^{-x} \right) \left( C \cos y + D \sin y \right) = 0 \]

Apply (iv)

\[ \frac{\partial}{\partial y} \left( A e^{x} + B e^{-x} \right) \left( C \cos y + D \sin y \right) = 0 \]

Most general solution:

\[ U(x,y) = f(x) g(y) + h(x) \]

Apply (v)

\[ U(0,y) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{\pi n y}{L} \right) \]

Apply (vi)

\[ U(x,0) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{\pi n x}{L} \right) \]

Most general solution:

\[ U(x,y) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{\pi n x}{L} \right) \sin \left( \frac{\pi n y}{L} \right) \]

Apply (vii)

\[ \frac{\partial}{\partial x} \left( A e^{x} + B e^{-x} \right) \left( C \cos y + D \sin y \right) = 0 \]

Apply (viii)

\[ \frac{\partial}{\partial y} \left( A e^{x} + B e^{-x} \right) \left( C \cos y + D \sin y \right) = 0 \]

Most general solution:

\[ U(x,y) = \sum_{n=0}^{\infty} a_n \sin \left( \frac{\pi n x}{L} \right) \sin \left( \frac{\pi n y}{L} \right) \]
Applications Of Partial Differential Equations

\[ u(x,y) = \sum_{n=1}^{\infty} \frac{-600 \left( \frac{\sin \left( \frac{n\pi x}{L} \right)}{n^2 \pi^2} + \frac{400}{n^2 \pi^2} \sin \left( \frac{n\pi y}{L} \right) \right)}{n^2 \pi^2} \sin \left( \frac{n\pi x}{L} \right) \]

\[ D_n = \frac{800}{n^2 \pi^2} \sin \left( \frac{n\pi x}{L} \right) \]

\[ u(x,y) = \sum_{n=1}^{\infty} \frac{-600 \left( \frac{\sin \left( \frac{n\pi x}{L} \right)}{n^2 \pi^2} + \frac{400}{n^2 \pi^2} \sin \left( \frac{n\pi y}{L} \right) \right)}{n^2 \pi^2} \sin \left( \frac{n\pi x}{L} \right) \]

A rectangular plate with insulated edges and 10 cm wide and 20 cm long, is placed in such a way that it may be considered to be infinite in length. The temperature on the short edge \( y = 0 \) is given by \( u = \frac{80}{100} (10 - x) \) and \( 5 \leq x \leq 10 \) for all other edges.

Find the steady state conduct.

The temperature is:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

The conduct are:

1. \( u(0,y) = 0 \)
2. \( u(x,0) = 0 \)
3. \( u(x,10) = 0 \)
4. \( u(x,0) = 0 \)
5. \( u(x,10) = 0 \)

The conduct are:

\[ u(x,y) = \left( \frac{a_0}{2} + b_0 \right) + \left( \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi y}{L} \right) \right) \right) \]

Apply boundary conditions.
Applications Of Partial Differential Equations

Apply \( \partial_t \) Condy. \( t=0 \)

\[ \begin{align*}
\phi &= B \sin (\pi y) (C e^{\pi x} + D e^{-\pi x}) \\
\psi &= B \sin \pi y \\
\sin \pi y &= \phi = \sin \pi y \\
P &= \pi y / 10
\end{align*} \]

Apply 5. \( \partial_t \)

\[ \begin{align*}
\phi &= B \sin (\pi n y) (C e^{\pi n x} + D e^{-\pi n x}) \\
\psi &= B \sin \pi n y \in C(\pi) \\
\psi &= = 0
\end{align*} \]

B Becomes.

\[ U(t, \pi y) = B \sin (\pi n y) e^{-n \pi y / 10} \]

The most gen. \( x = 20, y = 30 \)

\[ U(t, \pi y) = \sum_{n=1}^{\infty} \frac{B_n \sin (\pi n y)}{n} e^{-n \pi y / 10} \]

Apply 4. \( \partial_t \)

\[ \begin{align*}
U &= \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \frac{B_n \sin (\pi n y)}{n} e^{-n \pi y / 10} \\
D &= \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \frac{B_n \sin (\pi n y)}{n} e^{-n \pi y / 10}
\end{align*} \]

\[ \begin{align*}
D &= \frac{B_n}{n} \left[ \cos \left( \frac{n \pi x}{10} \right) + 100 \frac{n \pi x}{10} \sin \left( \frac{n \pi x}{10} \right) \right] + \\
&\quad \left[ -10 \left( 10 - n^2 \right) \frac{n \pi x}{10} \cos \left( \frac{n \pi x}{10} \right) - 100 \frac{n \pi x}{10} \sin \left( \frac{n \pi x}{10} \right) \right]
\end{align*} \]

\[ \begin{align*}
D &= \frac{800}{n^2 \pi^2} \sin (n \pi y) \\
\end{align*} \]
Applications Of Partial Differential Equations

\[ u(x,y) = \sum_{n=1}^{\infty} \frac{8}{n^2 \pi^2} \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{n \pi y}{a} \right) \]

Problems on Polar Co-ordinates.

A thin semi-circular plate of radius 'a' has its bounding diameter kept at temp. Zero and its circumference at k.

Find the temp. distribution in the steady state.

Solt:

Let the Centre of the Circle be Pole and the bounding diameter as the initial line.

The steady state temp. at any pt. \( p(x,\theta) \) be \( U(x,\theta) \) then \( U \) satisfies the eq.

\[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \theta^2} + \frac{U}{\rho^2} = 0 \]

The Cond.

1. \( u(x,0) = 0 \) in \( 0 \leq x \leq a \)
2. \( u(x,\pi) = 0 \) in \( 0 \leq x \leq a \)
3. \( u(a,\theta) = k \)

The suitable soln.

\[ U = (C_1 \cos \theta + C_2 \sin \theta + P) \left( C_3 \cos \theta + C_4 \sin \theta \right) \]

Applying cond in 1

\[ u(0,\theta) = (C_1 \cos \theta + C_2 \sin \theta + P) \theta = 0 \]

\[ C_1 \cos \theta + C_2 \sin \theta = 0 \]

\[ \Rightarrow C_3 = 0 \]

\[ \theta = 90^\circ \] in exp. 1

\[ U = (C_1 \cos \theta + C_2 \sin \theta) \left( C_4 \sin \theta \right) \]

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Applications Of Partial Differential Equations

\[ u(0, \pi) = \left( C_0 \delta^0 + C_2 \pi - \phi \right) \sin \pi n = 0 \]
\[ C_0 \delta^0 + C_2 \pi - \phi = 0 \]
\[ \sin \pi n = 0 \quad (\text{for } n = 0, 2, 4, \ldots) \]
\[ \sin \pi n = \begin{cases} 0 & \text{for } n = 0, 2, 4, \ldots \\ \sin n & \text{for } n = 1, 3, 5, \ldots \end{cases} \]
\[ u = C_0 \delta^0 + C_2 \pi \sin n \]
\[ u = C_0 \delta^0 \sin n \]

The most general solution is:

\[ u(r, \theta) = \sum_{n=1}^{\infty} C_n \sin n \theta \sin n \phi \]

Apply Cond. (3) in IV

\[ u(l, \theta) = \sum_{n=1}^{\infty} C_n \sin n \theta \sin n \phi \]

\[ \Rightarrow \sum_{n=1}^{\infty} b_n \sin n \phi = 0 \quad \text{where } b_n = C_n \phi \]

\[ b_n = \frac{2}{\pi} \int_0^\pi \sin n \phi \text{d} \phi \]

\[ = \frac{2}{\pi} \int_0^\pi \left( \cos n \phi \right) \text{d} \phi = \frac{2}{\pi} \int_0^\pi \cos n \phi \text{d} \phi \]

\[ = \frac{2}{n \pi} \left[ \sin n \phi \right]_0^\pi = \frac{2}{n \pi} \left[ \sin n \phi \right]_0^\pi \]

\[ C_n = \frac{b_n}{\sin n \phi} \]

\[ C_n = \begin{cases} 0 & \text{for } n = \text{even} \\ \frac{2}{an \pi} & \text{for } n = \text{odd} \end{cases} \]

\[ u(r, \theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{2}{an \pi} \sin n \phi \sin n \theta \]